Orsatti’s contribution to module theory

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I met Adalberto for the first time in November 1972: I was a first year student in Mathematics and he was my Algebra Professor. I do remember those tough days pretty well: the word “set” was completely new to me and each every time, at the end of Adalberto’s lesson, I thought I got an Arabic class instead of an Algebra one. Then suddenly, during Christmas time, I felt I could understand everything and that everything was so beautiful that, in the end, I gave up a very fancy interest for cybernetics (!) and decided to get more Algebra courses. Thus, in the next years, I followed two more courses by Adalberto and also got the honour to have him as my thesis advisor. Since then a never-ended collaboration began. Adalberto is not only a very clear teacher; he has the special gift of capturing your attention leading you inside the subject. For me he is the most fascinating person I ever met during my mathematical experience. And not only this. He taught me the most difficult thing: how to do research. It is nice here to remember that each every time I was getting depressed because we were not able to prove something then he began telling me one of his incredible jokes. Also if we found out that something we would have liked to be true, it was not, he was telling me “This is the truth, Claudia, what do you want more then the truth?”

Orsatti’s first work in module theory is his first paper on Duality [O1]. The astonishing idea he got was that Pontryagin duality between discrete and compact abelian groups (see [P1] and [P2]) and Lefschetz duality (see [L]) between discrete and linearly compact vector spaces as well as Kaplansky’s (see [K]) and Macdonald’s (see [Mc]) generalizations of this last one could be all thought as a particular case of a much more general situation. Namely, let $A$ be a commutative ring and $K$ an Hausdorff complete topological module over the ring $A$ endowed with its discrete topology. Assume that:

P$_1$) The mapping $a \mapsto \mu_a = \text{multiplication by } a$ in $K$ defines an isomorphism $A \cong \text{Chom}_A(K, K)$.

P$_2$) $K$ has no small submodules i.e. there is a neighbourhood $U$ of $0$ in $K$ such that the only submodule of $K$ contained in $U$ is $0$.

Let $A$-$\text{TM}$ denote the category of Hausdorff topological modules over the ring $A$ endowed with its discrete topology and let $C(K_A)$ be the subcategory of $A$-$\text{TM}$ consisting of those topological modules which are topological isomorphic to closed submodules of topological products of copies of $K$. Let $\mathcal{D}(K_A)$ be the subcategory of $\text{Mod}$-$A$ cogenerated by $K_A$. For every $M \in \text{Mod}$-$A$ let $M^*$ be the $A$-module $\text{Hom}_A(M, K)$ endowed with the topology of pointwise convergence. Clearly $M^* \in C(K_A)$. For every $M \in A$-$\text{TM}$, let $M^*$ be the abstract module $\text{Chom}_A(M, K)$;

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then $M^* \in \mathcal{D}(K_A)$. Having defined duals in this way, it is clear that the categories $\mathcal{D}(K_A)$ and $\mathcal{C}(K_A)$ are the largest one for which one can define duality functors. Thus let $\Delta_1: \mathcal{D}(K_A) \to \mathcal{C}(K_A)$ be the contravariant functor that associates to each $M \in \mathcal{D}(K_A)$ its dual module $M^*$ as defined above and let $\Delta_2: \mathcal{C}(K_A) \to \mathcal{D}(K_A)$ be defined in a similar way. $\Delta = (\Delta_1, \Delta_2)$ is a duality if every $M \in \mathcal{D}(K_A)$ and every $M \in \mathcal{C}(K_A)$ is reflexive i.e. the canonical morphism $\omega_M: M \to M^{**}$ is an isomorphism in the respective category. In this paper Orsatti characterized the dualities that are good i.e. those for which $\mathcal{C}(K_A)$ has the extension property of characters (i.e. every continuous morphism from a submodule $L$ of a module $M \in \mathcal{C}(K_A)$ into $K$ extends to a continuous morphism of $M$ into $K$) in the case that the topology of $K$ is compact or discrete. Then Pontryagin duality is recovered for $A = \mathbb{Z}$ and $K = \mathbb{R}/\mathbb{Z}$ while Macdonald duality is obtained with $A = \text{local}$, complete commutative noetherian ring and $K = \text{injective envelope of the unique simple } A\text{-module}$. In this paper Orsatti was wondering whether every duality was a good duality. Nowadays we know that the answer to this question is negative. In fact a counterexample in the discrete case was quickly given by Fuchs in [22], while the compact case remained open until when in [O9] dualities and equivalences met tilting modules. Fuchs not only solved the aforementioned question, but also pointed out the role of quasi-injectivity in connection with good dualities. In [O2], using Fuchs’ remarks, it is proved that, if $K$ is discrete and $R = \text{End}_R(K)$, $\Delta$ is a good duality if and only if $K$ is an s.q.i. (strongly quasi injective) module. A module $K$ is called strongly quasi-injective if for every submodule $B$ of $K$ and any $x \in K \setminus B$, each morphism $f: B \to K$ extends to an endomorphism of $K$ with $g(x) \neq 0$. Moreover if $R$ is any commutative ring, $R K$ is an s.q.i. $R$-module with commutative endomorphism ring and $A$ is the completion of $R$ in the $K$-topology, then $K$ has a natural structure of $A$-module, $\text{End}(K_A) = A$ and the $A$-module $K_A$ defines a good duality. Always in this paper the connections among duality, the theory of reflexive domains due to Matlis [M4] and the Gabriel filters with a basis of finitely generated projective ideals studied by Stenström in [S] are investigated.

The paper [O3] is perhaps the most quoted among Orsatti’s papers. Adalberto began to develop a duality theory for non commutative rings and it was at the beginning of 1980, when I came back from U.S.A. and I found myself involved with this more general business. Until then our experience with rings used to be restricted just with the commutative case. This was looking very natural as modules over commutative rings were somehow the closer subject to abelian groups and Adalberto’s previous interest in Algebra was just on this last topic. The impact of the new context was, at the beginning, somehow puzzling, at least for me. These coefficients that could not jump any longer from one side to the other and the module structure on $\text{Hom}(M, K)$ becoming more delicate were really something new to get used to. The new picture of the duality environment was the following. Let $R$ and $A$ be two rings (with identity $1 \neq 0$) and let $R K_A$ be a Hausdorff topological bimodule over $R$ and $A$ endowed with the discrete topology. Assume that $K_A$ is a faithful $A$-module and $K$ is complete in its canonical uniformity. Then $R$-$\text{TM}$, $\mathcal{D}(K_A)$ and $\mathcal{C}(R K)$ are defined in an analogous way as above. Also the definition of the dual functors is easily generalized to this more general situation and one gets a couple of contravariant functors $\Delta = (\Delta_1, \Delta_2)$ where $\Delta_1: \mathcal{D}(K_A) \to \mathcal{C}(R K)$ and $\Delta_2: \mathcal{C}(R K) \to \mathcal{D}(K_A)$. As $K_A$ is faithful, $A \in \mathcal{D}(K_A)$ and hence, if $\Delta$ is a duality, then $A = \text{Chom}_R(K, K)$. Therefore it is assumed that $A = \text{Chom}_R(K, K)$. Then if $K$ is discrete $\Delta$ is a good duality if and only if $R K$ is s.q.i., while if $K$ is compact then $\Delta$ is a good duality if and only if $R K$ is s.q.i. and has no small submodules. The other key result was the generalization to this general context of an idea of Bazzoni (see [3]). Let $R$-$\text{CM}$ denote the subcategory of $R$-$\text{TM}$
consisting of compact modules. Then for every \( M \in R\text{-CM} \) its Pontryagin dual has a natural structure of right \( R \)-module. Conversely if \( M \in \text{Mod-}R \) then its Pontryagin dual has a natural structure of left \( R \)-module and it belongs to \( R\text{-CM} \). Therefore Pontryagin duality induces a duality \( \Gamma = (\Gamma_1, \Gamma_2) \) where \( \Gamma_1: R\text{-CM} \to \text{Mod-}R \) and \( \Gamma_2: \text{Mod-}R \to R\text{-CM} \). Also, by the definition of \( \mathcal{C}(R,K) \), \( \Gamma \) induces a duality between \( \mathcal{C}(R,K) \) and the category \( \text{Gen}(P_R) \) of right \( R \)-modules generated by \( P_R = \Gamma_1(RK) \) and \( A = \text{End}_R(P_R) \). \( \Delta \) is a duality if and only if \( \Gamma \Delta = (\Gamma_1 \Delta_1, \Delta_2 \Gamma_2) \) is an equivalence between \( \mathcal{C}(R,K) \) and \( \text{Gen}(P_R) \). Moreover \( \Gamma_1 \Delta_1 \cong - \otimes_A P \) and \( \Delta_2 \Gamma_2 \cong \text{Hom}_R(P_R, -) \). Then the crucial result is that \( \Delta \) is a good duality if and only if \( - \otimes_A P \) is a category equivalence between \( \text{Mod-}A \) and \( \text{Gen}(P_R) \) and if only \( P_R \) is a quasi-progenerator in the sense of Fuller [F] (i.e. \( P_R \) is quasi-projective, finitely generated and generates all its submodules). Thus thanks to Bazzoni’s idea, the study of duality in the compact case was transformed into the study of equivalences. This was done in [06] and [09]. Let \( A \) and \( R \) be two rings, \( \mathcal{D}_A \) and \( \mathcal{G}_R \) subcategories of \( \text{Mod-}A \) and \( \text{Mod-}R \) respectively. Assume that a category equivalence \( (F,G), \mathcal{D}_A \to \mathcal{G}_R \) and \( G: \mathcal{G}_R \to \mathcal{D}_A \) is given and that \( A_A \in \mathcal{D}_A \), \( \mathcal{D}_A \) is closed under submodules and \( \mathcal{G}_R \) is closed under arbitrary direct sums and epimorphic images. Let \( Q_R \) be a fixed, but arbitrary, injective cogenerator of \( \text{Mod-}R \). Then this equivalence is representable i.e. there exists a bimodule \( A_P_R \), unique up to isomorphisms, such that \( F \) and \( G \) are naturally equivalent to the functors \( T = - \otimes_A P \) and \( H = \text{Hom}_R(P_R, -) \). Moreover \( \mathcal{D}_A = \mathcal{D}(K_A) \), where \( K_A = \text{Hom}_R(P_R, Q_R) \), and \( \mathcal{G}_R = \text{Gen}(P_R) \). In this case \( \text{Im}(T) \) and \( \text{Im}(H) \) are as large as possible. Then in [09] it is proved that if \( R \) is a ring, \( P_R \) is a \( w \)-tilting module (also called just tilting module in the literature) and \( A = \text{End}_R(P_R) \), then \( (T,H) \) is a category equivalence between \( \mathcal{D}_A = \{ L \in \text{Mod-}A : \text{Tor}_1^R(L_A, AP) = 0 \} \) and \( \text{Gen}(P_R) \). Let now \( P_R \) be a \( w \)-tilting non projective module (for an example see [HR], pp. 126-127). Then \( \mathcal{D}_A \neq \text{Mod-}A \) and setting \( R^A K_A = \Gamma_2(\mathcal{D}_A握住} \) we have \( R^A K_A \cong \text{Chom}_R(R_K, R^K) \). Moreover the couple of functors \( (\Delta_1, \Delta_2) \) defined by \( R^A K_A \) coincide with \( (HT_1, \Gamma_2 T) \) and \( \Delta = (\Delta_1, \Delta_2) \) is a duality between \( \mathcal{D}(K_A) = \mathcal{D}_A \neq \text{Mod-}A \) and \( \mathcal{C}(K_A) \). As \( P_R \cong \Gamma_1(K_R) \) we conclude that \( \Delta \) is a duality which is not a good duality. I would like to recall here that it was Masahisa Sato that pointed out to us that tilting modules might provide examples of \( \mathcal{D}(K_A) \neq \text{Mod-}A \). In fact Orsatti explained this problem to Sato during the N.A.T.O. meeting “Perspectives in Ring Theory”, that was held at Antwerpen (Belgium) in the period July 20–29 1987. After sometime Sato wrote to Orsatti showing an example of a tilting module \( P_R \) such that, for \( A = \text{End}(P_R) \) and \( K = \Gamma_2(A P_R) \), \( \mathcal{D}(K_A) \) was smaller than \( \text{Mod-}A \). Sato also thought that a tilting module \( P_R \) might have given rise to an equivalence between \( \mathcal{D}(K_A) \) and \( \text{Gen}(P_R) \). Always in [09] a representation theorem for dualities is given. The result quoted above on representation of equivalences suggested the following question—denoted by \( (*) \)—in ([09], 3.5)

\begin{equation}
(*) \text{ For a given ring } R \text{ determine all modules } P_R \in \text{Mod-}R \text{ such that setting } A = \text{End}(P_R), \text{ the bimodule } A P_R \text{ induces an equivalence between } \mathcal{D}(K_A) \text{ and } \text{Gen}(P_R). \end{equation}

It is nice to note that D’Este in [15] was the first one who called such modules \( * \)-modules, even if her motivation was of different typographic nature. The study of \( * \)-modules and their relations with tilting modules has been really booming (see the papers listed in the References about these subjects). In [44] Trlifaj solved one of the main problems by proving that every \( * \)-module is finitely generated. By a previous result of Colpi and Menini (see Theorem 2.4 in [7]), it follows that over a commutative ring \( R \) all \( * \)-modules are quasi-progenerators and hence, in this case, all dualities are good. Among the recent papers on this topic I would like to quote
Hopf-Galois extensions (see [Z]) was the starting point for applying module theoretical methods to the study of Huisgen [Z] is closed under submodules. Using some previous results due to Zimmermann-

$$\Delta: \text{Mod-}K \to \text{Mod-}K$$

is a left linear topology (such that $$\text{Mod-}K$$ is a left linearly compact ring). If $$R$$ has a quasi-duality if and only if $$R$$ has a good duality. This

$$\Delta: \text{Mod-}A \to \text{C}(R)$$

or, equivalently, such that $$K_A$$ is a cogenerator of $$\text{Mod-}A$$ and $$\text{C}(R)$$ is s.q.i.. This topic was further developed by Zelmanowitz and Jansen in [ZJ].

The work [O10] is a topological investigation on rings having a quasi-duality. A ring $$R$$ has a left quasi-duality in the sense of Kraemer [KR] if there is a ring $$A$$ and a faithfully balanced bimodule $$RKA$$ such that both $$RKA$$ and $$KA$$ are quasi-injective and finitely cogenerated. A bimodule of this type is called a GM-bimodule. Then in [O10] it is proved that GM-bimodules are exactly those that yield a duality between suitable categories of linearly topologized modules over the linearly topologized rings $$A$$ and $$R$$. This generalizes Müller’s main Theorem in [Mu3]. Moreover it is proved that a ring $$R$$ has a left quasi-duality if and only if $$R$$ is semiperfect and there is a left linear topology $$\tau$$ on $$R$$ such that $$(R, \tau)$$ is linearly compact; in this case if $$R$$ is commutative, then it has a quasi-duality with itself. Using these results the well known Müller’s Theorem 1 and Theorem 2 in [Mu1] are obtained as corollaries. In [25] Gregorio proved that right quasi-duality is a Morita invariant.

In [O5] dualities between categories of topological modules are considered. Let $$A$$ and $$R$$ be discrete rings and let $$H = (H_1, H_2)$$ be a duality between a full subcategory $$B_A$$ of $$\text{TM-}A$$ (here the objects of $$\text{TM-}A$$ are not assumed to be Hausdorff) and a full subcategory $$R\mathcal{B}$$ of $$\text{R-TM}$$. Under very natural assumptions on the categories $$B_A$$ and $$R\mathcal{B}$$ it is proved that there is a topological bimodule $$RKA$$ (such that $$RKA$$ is $$B_A$$ of $$\text{R-TM}$$ with two possibly different topologies) that induces a duality between the categories $$B_A$$ and $$R\mathcal{B}$$ which are obtained by substituting the topology of each module in $$B_A$$ and $$R\mathcal{B}$$ with the weak topology of its $$K$$-characters. Moreover if $$B_A$$ and $$R\mathcal{B}$$ are closed under topological products and submodules then the categories $$\mathcal{B}_A$$ and $$\mathcal{B}_A$$ are as large as possible i.e. $$\mathcal{B}_A = \mathcal{B}(K_A)$$ and $$\mathcal{B} = \mathcal{B}(\mathcal{K})$$ where $$\mathcal{B}(K_A)$$ is the category of $$K_A$$-completely regular modules, i.e. of those bimodules which are topologically isomorphic to submodules of topological powers of $$K_A$$ and $$\mathcal{B}(\mathcal{K})$$ is the category of $$\mathcal{K}$$-completely regular modules. When the modules of $$B_A$$ and $$R\mathcal{B}$$ are discrete then this result gives back the classical Morita
Theorem 6.6 in [O5] pseudo-dualities between the categories of linearly topologized Hausdorff modules over the rings $R$ and $A$ endowed with linear Hausdorff topologies are studied obtaining a result (see Theorem 6.6 in [O5]) which considerably generalizes Müller’s Duality Theorem in [Mu2]. The relations between the duality $D$ and the duality $\Delta$ are also investigated.

The extensive work [O6] is a general treatment of Pontryagin-Morita type dualities between categories of topological modules. The first four Chapters present a unified and extensive study of the dualities $D$ and $\Delta$. In this integration procedure new results are found. An example of a faithfully balanced topological bimodule $R_K$ endowed with two different topologies $\chi$ and $\chi'$ such that both $(K, \chi)$ and $(R_K, \chi')$ are compact, q.i. and without small submodules is given. Other examples can be found in [36]. Also in [36] Preciso characterized faithfully balanced topological bimodule $R_K$ such that $\chi$ is the discrete topology, $K$ is quasi-injective, $\chi'$ is compact and $(R_K, \chi')$ is quasi-injective and has no small submodules showing that, in this case, $K$, $A$ and $R$ must be finite. The last three Chapters in [O6] are devoted to s.q.i. discrete modules, linear compact rings and their dualities; in particular Anh’s Topological Morita Duality (see [A1]) is considered. As an application classical Müller’s results on Morita Duality (see [Mu1] and [Mu2]) are obtained.

Let $(R, \tau)$ be a ring endowed with a left linear Hausdorff topology $\tau$. The topological ring $(R, \tau)$ is a topologically artinian ring (TA-ring) if the module $R/I$ is artinian for every open left ideal $I$. The concept of TA-ring was introduced by Ballet [B], who mostly dealt with commutative TA-rings. The paper [O7] is devoted to the study of TA-rings. Let $RU$ be the minimal cogenerator of the class of $\tau$-torsion modules, $A = \text{End}(RU)$ and let $\hat{R}$ be the Hausdorff completion of $(R, \tau)$. Assume that $(R, \tau)$ is a TA-ring. Then the bimodule $RU_A$ is faithfully balanced and the module $U_A$ is s.q.i.. Moreover the class of $\tau$-torsion modules is contained in the hereditary torsion class generated by the class of simple $\tau$-torsion left $R$-modules. Let $\tau_\Omega$ be the topology on $R$ having a basis of neighbourhoods of $0$ the closures $J^n$ in $(R, \tau)$ of $J^n$, where $J$ is the topological Jacobson radical of $R$. If $(R, \tau)$ is a topological semilocal ring and $J$ is finitely generated then $(R, \tau)$ is a TA-ring iff $\tau = \tau_\Omega$ iff $\tau$ coincides with its Leptin topology and $RU = \bigcup_{n \in \mathbb{N}} s_n(U)$ where $s_n$ is the $n$-th socle. Some other results are proved showing that this type of TA-rings has properties analogous to those of commutative local noetherian rings.

Then the topological Artin-Rees property for $J$ is introduced and related with some noetherian type properties of $R$. Finally TA-rings $(R, \chi$) where $\chi$ is the cofinite topology of $R$ are studied and the following question is asked: “Is it true that, for every open left ideal $I$ of $R$, $R/I$ has finite length?” A negative answer to this question is given in [15] and in [34].

In the paper [O8] the notion of the basic ring of a locally Artinian Grothendieck category $\mathcal{C}$ is introduced. Namely let $W$ be the minimal cogenerator of $\mathcal{C}$, let $\mathcal{G}$ be the subcategory of $\mathcal{C}$ consisting of all subobjects of the finitely generated objects
in \( \mathcal{C} \) and let \( A \) be the ring \( \text{End}_\mathcal{C}(W) \) endowed with the topology \( \sigma \) having all the annihilators in \( A \) of the subobjects \( N \in \mathfrak{N} \) of \( W \) as a basis of neighbourhoods of 0. Let \( K_A \) be the minimal injective cogenerator of the class \( \mathfrak{T}_\sigma \) of all \( \sigma \)-torsion right \( A \)-modules. The basic ring of \( \mathcal{C} \) is the topological ring \( (B, \beta) \) where \( B = \text{End}(K_A) \) and \( \beta \) is the \( K \)-topology of \( B \). Then, using Oberst Duality for Grothendieck Categories [Ob], some important properties of linearly compact rings and the non trivial fact that the above subcategory \( \mathfrak{N} \) of \( \mathcal{C} \) generates \( \mathcal{C} \), it is proved that: (a) \( (B, \beta) \) is strictly linearly compact and \( B/J(B) \) is a direct of division rings; (b) the category \( \mathcal{C} \) is equivalent to the category \( \mathfrak{T}_\beta \) of all \( \beta \)-torsion left \( B \)-modules; (c) two locally artinian Grothendieck categories are equivalent if and only if their basic rings are topologically isomorphic. Moreover using the basic ring some improvements of results by Ánh [A2] and Năstăcescu [N1], [N2] are obtained. A locally artinian Grothendieck category \( \mathcal{C} \) is commutative if \( \mathcal{C} \) admits a generator with commutative endomorphism ring. In [O11] it is proved that the basic ring of a commutative locally artinian Grothendieck category is the topological product of commutative local artinian rings.

The book [O14] is a nice introduction to some topics in Module Theory. It includes, besides some classical and basic subjects, a chapter on linearly compact rings and modules, one on structure theorems of linearly compact rings, one on the endomorphism ring of an infinite dimensional vector space, one on equivalences and one on dualities. It is a new useful tool for a course on these subjects.

The References below are subdivided in three parts: the first one is the list of Orsatti’s papers that were the subject of this talk, the second one includes the other works that were cited above and that do not appear in the third one and finally the third one is a list of papers that are somehow related to Orsatti’s contribution to Module Theory. I have tried to make this list as exhaustive as possible (I apologize for anybody’s papers I forgot to mention). In fact I do believe that this shows how fruitful are Orsatti’s ideas. It also shows that one of his major contributions to the Theory is to have a considerable group of “young” researchers involved with this subject. My personal wish is that Adalberto goes on making this group larger and larger.

References


