Categorical equivalences and realization theorems

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Abstract

We demonstrate an equivalence between general types of Grothendieck categories and specific subcategories of the category of modules over certain endomorphism rings. This will yield as corollaries the equivalence results of Cohen–Montgomery and del Río. In addition, this equivalence also yields information about the graded modules over rings graded by categories.

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0. Introduction

Categories of graded objects which arise in the investigation of group-graded rings have been explicitly described as module categories over various “smash product” rings. Examples of such “realization” results include those of Cohen and Montgomery [6, Theorem 2.2], Beattie [5, Theorem 2.6], Albu [3, Proposition 1.8], and del Río [10, Corollary 3.6]. The main contribution of this article is Theorem 1.9, which presents a unified approach to these results. We then show in Section 2 that there is a large class of rings (called “rings graded by categories”), which include group-graded rings, to which this realization theorem applies.

Historically, the aforementioned results arose as follows. In the fundamental paper [6], Cohen and Montgomery showed (among other things) that if $R$ is a $k$-algebra graded by the finite group $G$, then the category $R$-$gr$ of $G$-graded left $R$-modules is equivalent to the full category of left modules over the smash product ring $R#k[G]$.*

Many different approaches were taken to produce an appropriate generalization of this result for infinite groups. Beattie [5] produced a (nonunital) smash product ring whose
full category of unitary modules is equivalent to $R$-$gr$ for infinite $G$. Quinn [9] investigated a unital smash product ring for infinite $G$ which retained the germane fixed ring and skew group ring properties of the finite case. Albu and Năstănescu [4] extended the Cohen–Montgomery result with an eye towards describing the relationships between Quinn's smash product ring and the ring $\text{End}_{R-gr}(U)$ of graded endomorphisms of a canonical generator $U$ of $R$-$gr$. The Albu–Năstănescu endomorphism ring, Quinn's smash product ring, and Beattie's smash product ring each coincide in the finite case with the Cohen–Montgomery construction; thus it is perhaps surprising that these three generalizations of [6] to infinite groups led to three distinct types of infinite matrix rings. In [7], the second author obtained realization results for a large class of rings (which includes the aforementioned three specific types) by utilizing the Gabriel–Popescu theorem.

Recently, Albu [3] and del Río [10] have shown that these categories of graded modules can be realized as subcategories of modules over appropriate matrix rings generated by a canonical module, with del Río's results being cast in the more general setting of $G$-sets.

The purpose of the current article is twofold. First, we show that the Gabriel–Popescu result may be recast to yield more information than described in [7]. This new result yields, in the case of $(G,X,R)$-$gr$, a categorical version of del Río's result [10, Corollary 3.6] which explains how any of the appropriate infinite matrix rings (including those investigated by Beattie, Quinn, and Albu–Năstănescu mentioned above) may be used to realize the category $R$-$gr$. Secondly, we show that such results may be applied to structures more general than group-graded structures (so-called "rings graded by categories"); we thereby obtain each of the aforementioned results as a corollary.

In this article we will discuss two types of categories; the symbol $\mathcal{C}$ will denote a small category, while the symbol $\mathcal{G}$ will be used to denote a Grothendieck category. The collection of objects in a category will be denoted by $\text{Ob}(\mathcal{C})$ and the collection of morphisms by $\mathcal{M}(\mathcal{C})$. If $f : B \to B'$ is a morphism in a category, then $B$ (resp. $B'$) will be denoted by $D(f)$ (resp. $C(f)$). For each object $B$ in a category, the identity morphism on $B$ will be denoted by $1_B$. Unless otherwise indicated, all morphisms will be composed from left to right (so that the notation $fg = f \circ g$ will mean "first $f, then $g")$. All rings are assumed to be associative, possibly without identity (unless otherwise indicated).

1. The equivalence theorem

The goal of this first section is to prove the "realization" theorem described in the introduction; we accomplish this in Theorem 1.9.

Let $(U_x)_{x \in X}$ be a family of objects of the Grothendieck category $\mathcal{G}$, $\hat{U} = \prod_{x \in X} U_x$ their product, and $U = \coprod_{x \in X} U_x$ their coproduct. Let $j : U \to \hat{U}$ denote the canonical injection. For every $x \in X$ let $e_x : U_x \to U$ and $e_x = e_x j : U_x \to \hat{U}$ denote the canonical injections and $\pi_x : \hat{U} \to U$ and $p_x = j \pi_x : U \to U_x$ denote the canonical projections. Let
\[ \eta_x = \pi_x e_x, \quad h_x = p_x e_x \text{ and } \ell_x = \pi_x e_x. \]  
\[ P_0(X) \]  
will denote the set of finite subsets of \( X \).

Given \( F \neq \emptyset \) in \( P_0(X) \) we set \( \eta_F = \sum_{x \in F} \eta_x \); we define \( \eta_{\emptyset} = 0 \). Given \( F, G \in P_0(X) \), we have

\[
\eta_F + \eta_G = \eta_{F \cup G} \quad \text{(for } F \cap G = \emptyset), \quad \text{and} \quad \eta_F \cdot \eta_G = \eta_{F \cap G} = \eta_G \cdot \eta_F.
\]

In particular, \( \eta_F \eta_G = \eta_F \) if and only if \( G \supseteq F \). We let \( \hat{A} \) denote the ring

\[
\hat{A} = \text{End}_{\mathfrak{q}}(\hat{U});
\]

then \( \hat{A} \) is a unital ring with respect to the multiplication \( \alpha \beta = \alpha \circ \beta \) (for \( \alpha, \beta \in \hat{A} \)). We denote by \( A^0 \) the set

\[
A^0 = \left\{ \alpha \in \hat{A} \mid \forall x \in X, \exists F \in P_0(X) \text{ such that } \alpha \eta_x = \eta_F \alpha \eta_x \right\}.
\]

Given \( \alpha \in A^0 \) and \( x \in X \) we set

\[
F^x = \bigcap \left\{ F \in P_0(X) \mid \eta_F \alpha \eta_x = \alpha \eta_x \right\}.
\]

Note that \( \eta_{F^x} \alpha \eta_x = \alpha \eta_x \). In fact, given \( F \in P_0(X) \) such that \( \eta_F \alpha \eta_x = \alpha \eta_x \) we have that

\[
F^x = \bigcap \left\{ G \in P(F) \mid \eta_G \alpha \eta_x = \alpha \eta_x \right\}.
\]

**Lemma 1.1.** Let \( \alpha \in A^0, G \in P_0(X), F = \bigcup_{x \in G} F^x \). Then \( \alpha \eta_G = \eta_F \alpha \eta_G \).

**Proof.** We have the following sequence of equalities:

\[
\alpha \eta_G = \sum_{x \in G} \alpha \eta_x = \sum_{x \in G} \eta_{F^x} \alpha \eta_x = \sum_{x \in G} \eta_F \eta_{F^x} \alpha \eta_x = \eta_F \cdot \sum_{x \in G} \eta_{F^x} \alpha \eta_x = \eta_F \alpha \eta_G. \quad \square
\]

We denote by \( I \) the set

\[
I = \left\{ \eta_F fj \mid F \in P_0(X), f \in \text{Hom}_{\mathfrak{q}}(\hat{U}, U) \right\}.
\]

**Proposition 1.2.** \( A^0 \) is a unital subring of \( \hat{A} \), \( I \) is an idempotent left ideal of \( A^0 \), and

\[
\bigoplus_{x \in X} A^0 \eta_x \subseteq I.
\]

Moreover, equality holds in the above inclusion if and only if each \( U_x \) is small.

**Proof.** Let \( \alpha, \beta \in A^0 \) and let \( x \in X \). Then \( \eta_{F^x \cup \beta F^x} (\alpha + \beta) \eta_x = (\alpha + \beta) \eta_x \). Moreover, setting \( G = \bigcup_{y \in F^x} F_Y^\beta \) we have, by Lemma 1.1, \( \eta_G \beta \alpha \eta_x = \eta_G \beta \eta_{F^x} \alpha \eta_x = \beta \eta_{F^x} \alpha \eta_x = \beta \alpha \eta_x \).

Thus, \( A^0 \) is a unital subring of \( \hat{A} \).

It is easy to prove that \( I \) is closed under addition. Now let \( \alpha \in A^0, G \in P_0(X), f \in \text{Hom}_{\mathfrak{q}}(\hat{U}, U) \). Again using Lemma 1.1 we get \( \alpha \eta_G fj = \eta_F \alpha \eta_G fj \) for \( F = \bigcup_{x \in G} F^x \), so that \( I \) is a left ideal of \( A^0 \). Given \( x \in X \) we have \( \eta_x = \eta_x \eta_x = \eta_x \ell_x j \in I \). Hence by
the foregoing, $\bigoplus_{x \in X} A^0\eta_x = \sum_{x \in X} A^0\eta_x \subseteq I$. Moreover, as every $\eta_F$ (for $F \in P_0(X)$) belongs to $I$, $I$ is an idempotent left ideal.

Assume now that each $U_x$ is small. Then, given $x \in X$ and $f \in \text{Hom}_q(U, U)$ there exists $F \in P_0(X)$ such that $e_x f = e_x F \cdot \sum_{y \in F} h_y$. Therefore, we get

$$\eta_x f = \pi_x e_x f = \pi_x e_x \left( \sum_{y \in F} h_y \right) = \eta_x f \eta_F \in \sum_{y \in F} A^0\eta_y.$$ 

Conversely, assume that $I \subseteq \sum_{x \in X} A^0\eta_x$. Let $x \in X$, $f \in \text{Hom}_q(U_x, U_x)$ and set $g = \pi_x f$. Then $I$ contains $\eta_x g = \pi_x e_x \pi_x f j = g j$, so that $g j = \sum_{y \in F} \alpha_y \eta_y$ for suitable $F \in P_0(X)$ and $\alpha_y \in A^0$. It follows that

$$f j = e_x \pi_x f j = e_x g j = e_x \cdot \sum_{y \in F} \alpha_y \pi_y e_y j.$$ 

Therefore, $f = e_x \cdot \sum_{y \in F} \alpha_y \pi_y e_y$, and $U_x$ is small. \hfill $\square$

**Definition 1.3.** We denote by $H(U)$ the ring of all $X$-square column finite matrices $A$ such that, for each $(x, y) \in X \times X$, the entry in the $(x, y)$ position of $A$ (denoted by $A_{x,y}$) belongs to $\text{Hom}_q(U_x, U_y)$. For $A, A' \in H(U)$ the product $A A'$ is given by

$$(A A')_{x,y} = \sum_{z \in X} A_{x,z} A'_{z,y}.$$ 

The importance of $H(U)$ in the sequel is twofold. First, it is straightforward to show that the mapping $\alpha \mapsto (e_x \alpha \pi_y)_{x \in X, y}$ gives a ring isomorphism between $A^0$ and $H(U)$. In fact, given $A \in H(U)$ we have $A_{x,y} = e_x \alpha \pi_y$, where $\alpha$ denotes the diagonal map of $(\sum_{x \in X} \pi_x A_{x,y})_{y \in X}$. Thus $H(U)$ affords a concrete, matricial description of the ring $A^0$. In addition, it is easy to check that if each $U_x$ is small, the image of $I$ under this isomorphism is

$$I(U) = \{ A \in H(U) \mid A_{x,y} = 0 \text{ for almost every } (x, y) \in X \times X \}.$$ 

Secondly, for each $M \in \text{Ob}(\mathcal{C})$ we define

$$Hq(\mathcal{U}, M) = \{ \eta_F f \mid F \in P_0(X), f \in \text{Hom}_q(U, M) \}.$$ 

Given $A \in A^0$ and $G \in P_0(X)$ we have by Lemma 1.1 that $\alpha \eta_G = \eta_F \alpha F \eta_G$ for $F = \bigcup_{x \in G} F_x$. Therefore, $Hq(\mathcal{U}, M)$ has the natural structure of a left $A^0$-module. Motivated by similar constructions made by Beattie and Albu–Năstăsescu, we will view $Hq(\mathcal{U}, M)$ as certain column vectors (indexed by $X$) having at most finitely many nonzero entries. From this perspective we see that $H(U)$ (and hence $A^0$) is the largest possible infinite matrix ring which may support $Hq(\mathcal{U}, M)$ as a left module. Specifically, the finite column matrices $Q$ are precisely those with the property that for each column vector $Z$ having finitely many entries, $QZ$ again has finitely many entries.

With the above discussion in mind, for the remainder of this article we will denote by $A$ any arbitrary (but fixed) subring of $A^0$ containing $I$. We do not assume
that $A$ is unital, so that (for instance) we allow the possibility that $A=I$. Moreover, $H^q_A(\hat{U}, M)$ will denote the abelian group $H^q_\mathcal{E}(\hat{U}, M)$ equipped with the left $A$-module structure induced by the $A^0$-module structure described above. For a given $A$, $A$-mod will denote the category of all left $A$-modules and $A$-Mod will denote the category of all unitary left $A$-modules (i.e., $A$-modules $M$ for which $AM=M$). For $F \in P_0(X)$ and $f \in Hom_{\mathcal{E}}(\hat{U}, U)$ the assignment $\eta_F f \mapsto \eta_F f j$ yields an isomorphism of $H^q_\mathcal{E}(\hat{U}, U)$ into $I$. Clearly, the mapping $M \mapsto H^q_A(\hat{U}, M)$ defines a left exact covariant functor

$$H^q_A(\hat{U}, -): \mathcal{E} \to A\text{-mod}.$$  

We note here that $H^1_A(\hat{U}, -)$ is isomorphic to the functor $\overline{\text{Hom}}_\mathcal{E}(U, -)$ defined in [7].

**Proposition 1.4.** (a) If each $U_x (x \in X)$ is small, then the functor $H^q_A(\hat{U}, -)$ commutes with coproducts.

(b) If each $U_x (x \in X)$ is projective, then the functor $H^q_A(\hat{U}, -)$ is exact.

(c) If $(U_x)_{x \in X}$ is a system of generators of $\mathcal{E}$, then the functor $H^q_A(\hat{U}, -)$ is full and faithful.

**Proof.** Parts (a) and (b) are straightforward. It is not hard to verify (using the supposed inclusion $I \subseteq A$) that the proof of [7, Theorem 2.6] carries over verbatim to this more general setting, which gives (c). \qed

**Definition 1.5.** Given a left $A$-module $M$ we let $Gen(A M)$ denote the full subcategory of $A$-mod formed by the left $A$-modules $X$ generated by $M$; that is, modules for which there exists an exact sequence of the form $M^{(T)} \to X \to 0$.

An easy calculation shows that if $A M \in A$-Mod, then $Gen(A M) \subseteq A$-Mod.

**Lemma 1.6.** For every $M \in Ob(\mathcal{E})$, $H^q_A(\hat{U}, M) \in Gen(A I)$. Moreover,

$$Gen(A I) = \{ L \in A\text{-mod} \mid IL = L \} = \left\{ L \in A\text{-mod} \mid L = \bigoplus_{x \in X} \eta_x L \right\}.$$

In particular, $Gen(I I) = I$-Mod.

**Proof.** Given $F \in P_0(X)$ and $f \in Hom_{\mathcal{E}}(\hat{U}, M)$ we have $\eta_F f = \eta_F (\eta_F f)$. Hence, $H^q_A(\hat{U}, M) \in Gen(A I)$. The remaining statements follow from the fact that $I$ is idempotent. \qed

**Lemma 1.7.** The "restriction of scalars" functor $D: Gen(A I) \to Gen(I I) = I$-Mod is a category equivalence.

**Proof.** This proof is modeled on that of [10, Lemma 3.5]; we include the details for completeness. The functor $D$ is obviously faithful. Let $M, N \in Gen(A I)$ and $f \in$
Given \( m \in M = IM \) there is an \( F \in P_0(X) \) such that \( m = \eta_F m \). Let \( a \in A \). Then

\[(am)f = (a\eta_F m)f = a \eta_F \cdot (\eta_F m)f = a \cdot (\eta_F m)f = a \cdot (m)f\]

with the second equality holding as \( \eta_F \in I \). Therefore, \( f \in Hom_A(M,N) \), so that \( D \) is full.

Now let \( M \in Gen(I) \). As above, for \( m \in M = IM \) we have \( m = \eta_F m \) for some \( F \in P_0(X) \). For each \( a \in A \) we set

\[a \cdot m = (\eta_F)m.\]

We must show this definition does not depend on \( F \). To this end let \( G \in P_0(X) \) with \( m = \eta_G m \). Then \( (\eta_F)m = (\eta_{FUG})\eta_f m = (\eta_{FUG})\eta_f m = (\eta_{FUG})\eta_G m = (\eta_{FUG})\eta_G m = (\eta_{FUG})\eta_G m = (\eta_G)m \). Therefore the above definition yields a left \( A \)-module structure on \( M \). As \( D \) is full we get \( _AM \in Gen_0(I) \), and \( D(M) = M \). \( \square \)

We are now in a position to prove the main results of this article.

**Theorem 1.8.** Let \( (U_x)_{x \in X} \) be a system of small generators of a Grothendieck category \( \mathcal{C} \). Then the functor

\[S = H^0_G(\mathcal{U},-): \mathcal{C} \to Gen(A)\]

has a left adjoint \( T \). Moreover, the adjunction \( TS \to 1_\mathcal{C} \) is an isomorphism and \( T \) is exact. Thus, \( S \) induces an equivalence between \( \mathcal{C} \) and the quotient category of \( Gen(A) \) by \( \text{Ker}(T) \).

**Proof.** By [7, Theorem 2.11] the result is valid for the particular functor \( S = H^0_G(\mathcal{U},-). \)

We now need only apply Lemma 1.7 to conclude that the equivalence holds for all functors of the specified type. \( \square \)

In a manner analogous to the above theorem, we similarly obtain

**Theorem 1.9.** Let \( (U_x)_{x \in X} \) be a system of small projective generators of a Grothendieck category \( \mathcal{C} \), and let \( A \) be any subring of \( A^0 \) containing \( I \). Then the functor

\[S = H^0_G(\mathcal{U},-): \mathcal{C} \to Gen(A)\]

is an equivalence.

We conclude this section by pointing out the connection between our results and those of del Rio [10]. Let \( G \) be a group and let \( X \) be a left \( G \)-set. Given a \( G \)-graded ring \( R \), \((G,X,R)-gr\) will denote the category of (left) graded \( R \)-modules of type \( X \); i.e., of those \( R \)-modules \( M \) such that \( M = \bigoplus_{x \in X} M_x \) as additive subgroups, and for all \( g \in G, x \in X \) we have \( R_g M_x \subseteq M_{gx} \). Let \( B^0 \) be the subring of the ring \( FCM_X(R) \).
of $X$-square column finite matrices over $R$ defined by

$$B^0 = \left\{ A \in FCM_X(R) \mid A_{x,y} \in \sum_{y=x} R_g \right\}.$$  

Since the collection $\{R(x) \mid x \in X\}$ of $x$-suspensions form a system of finitely generated projective generators of $(G,X,R)$-$gr$ (see [8, Theorem 2.8]), and since

$$\text{Hom}_{(G,X,R)}(R(x), R(y)) \cong \bigoplus_{y=x} R_g$$

via right multiplication, we get as a corollary of Theorem 1.9 the following result of del Rio (which in turn gives the aforementioned realization results of Cohen–Montgomery, Beattie, Albu–Năstăscu, and Albu).

**Corollary 1.10** (del Rio [10, Corollary 3.6]). For every subring $B$ of $B^0$ which contains $I = \{A \in B^0 \mid A_{x,y} = 0 \text{ for almost every } (x, y) \in X \times X\}$, there is an equivalence of categories between $(G,X,R)$-$gr$ and $\text{Gen}(B)$.

We will describe in the next section another class of graded structures to which Theorem 1.9 may be applied.

### 2. Rings graded by categories

If $G$ is a group, then $G$ may be viewed as a category having one object, whose morphisms are the elements of $G$, and with composition of morphisms defined to be multiplication in $G$. From this point of view, the definition below follows as a natural generalization.

**Definition 2.1.** Let $C$ be any small category (i.e., the collection of morphisms $M(C)$ is a set). We say that a ring $R$ is *graded by the category $C$* if there is a family $\{R_f \mid f \in M(C)\}$ of additive subgroups of $R$ such that $R = \bigoplus_{f \in M(C)} R_f$, and for each pair $f, g \in M(C)$ we have

$$R_f \cdot R_g \begin{cases} \subseteq R_{fg} & \text{if } fg \text{ is defined,} \\ 0 & \text{otherwise.} \end{cases}$$

If $R$ is a ring graded by the category $C$, we say that the unitary left $R$-module $N$ is *graded by $C$* if there is a family $\{N_f \mid f \in M(C)\}$ of additive subgroups of $N$ such that $N = \bigoplus_{f \in M(C)} N_f$, and $R_f N_g \subseteq N_{fg}$ whenever $fg$ is defined, zero otherwise. We say that the $R$-linear map $A : N \to N'$ is a *graded morphism* if $(N_g)A \subseteq N'_{fg}$ for all $g \in M(C)$; we denote the collection of graded morphisms from $N$ to $N'$ by $\text{Hom}_{R-gr}(N,N')$. We will denote by $R-gr$ the category of graded left $R$-modules and graded morphisms.
Many well-known classes of rings are examples of rings graded by categories. As a first example, by viewing a group as a category as described above it is easy to see that group-graded rings and modules are objects graded by a category.

If \((X, \leq)\) is a preordered set, we may view \((X, \leq)\) as a category whose object set is \(X\), with morphisms \(\text{Mor}_X(x, y) = \{\leq_{xy}\}\) when \(x \leq y\), and \(\text{Mor}_X(x, y) = \emptyset\) otherwise. Composition is given by \(\leq_{xy} \circ \leq_{yz} = \leq_{xz}\); note that \(\leq_{xz}\) is a morphism by transitivity. If \(X\) is locally finite and \(A\) is any unital ring, then the incidence ring \(I(X, A)\) is a unital ring which is graded by the category \(X\).

As another example, let \(A\) be any unital ring, and let \(X = \{w, x, y, z\}\) denote the partially ordered set whose relations are given by \(\{w \leq x \leq z, w \leq y \leq z\}\). Let \(C\) denote the partially ordered set \(\{a, b\}\) where \(a \leq b\). Then \(M(C)\) consists of the three morphisms \(\{\leq_{aa}, \leq_{ab}, \leq_{bb}\}\), which we denote by \(1, \alpha, \beta\), respectively. A straightforward computation shows that the incidence ring \(R = I(X, A)\) is graded by \(C\) as follows (where \(e_{a,b}\) denotes the standard matrix unit in the \(a\)-row, \(b\)-column):

\[
R_1 = Ae_{w,w}, \quad R_2 = Ae_{x,x} \oplus Ae_{y,y} \oplus Ae_{z,z} \oplus Ae_{x,y} \oplus Ae_{z,x} \oplus Ae_{y,z}.
\]

If \(\Gamma\) is a quiver then \(\Gamma\) may be viewed as a category whose object set is the vertices of \(\Gamma\), and \(\text{Mor}_\Gamma(v, w) = \{\text{directed paths which originate at } v \text{ and terminate at } w\}\). (For each vertex \(v\) in \(\Gamma\) we include the trivial path inside \(\text{Mor}(v, v)\).) Composition is given by juxtaposition. For any field \(k\) and any quiver \(\Gamma\) the path algebra \(k\Gamma\) is a ring graded by the category \(\Gamma\). We note that \(k\Gamma\) is unital if and only if \(\Gamma\) has finitely many vertices. More generally, if \(I\) is an ideal of \(k\Gamma\) generated by paths, then the quotient \(k\Gamma/I\) (a so-called "monomial algebra") is graded by \(\Gamma\).

These and other examples of rings graded by categories are described in further detail in [1].

**Definition 2.2.** Suppose that \(R\) is a ring graded by the category \(C\), and suppose \(N \in R\text{-gr}\). Given \(f, g \in M(C)\) we set

\[
N(f)_g = \sum \{N_h \mid h \circ f \text{ is defined and } h \circ f = g\}.
\]

It is trivial to check that the sum

\[
\sum \{N(f)_g \mid g \in M(C)\}
\]

is direct; we denote it by \(N(f)\). Since \(R_k \cdot N(f)_g \subseteq N(f)_{kg}\) if \(k, g \in M(C)\) with \(kg\) defined, and is zero otherwise, we see that \(N(f)\) is in fact a graded \(R\)-module.

We may associate with any small category \(C\) the semigroup \(S_C = M(C) \cup \{z_C\}\) (where \(z_C\) is any symbol not in \(M(C)\)), having the same compositions as in \(M(C)\), but with \(f \cdot g = z_C\) in \(S_C\) whenever \(f \circ g\) is not defined in \(M(C)\). With this in mind,
one might expect that all of the above definitions and constructions would be valid in this more general setting of semigroups with zero. However, the following issue arises: if \( z \) denotes the zero element in the semigroup \( S \), what should the definition of \( N(f)_z \) be?

If we define \( N(f)_z = \sum_{h \in S} \{N_h \mid hf = z\} \), then if \( S \) is not of the form \( SC \) for some category \( C \) it is in fact possible to have \( N(f)_z \neq \{0\} \) while \( N_z = \{0\} \). In particular, if the goal is to study \( S \)-graded modules having \( \{0\} \) as the \( z \)-component, then this definition of \( N(f)_z \) is not appropriate in non-categorical settings.

On the other hand; if we attempt to rectify the problem posed in the previous paragraph by studying \( S \)-graded modules having no restriction on the \( z \)-component, then unfortunately we do not get that the set \( \{R(f) \mid f \in S\} \) is a family of projective generators for \( R_{-gr} \). As this set is of fundamental importance in our discussion (cf. Proposition 2.7), we do not in this article investigate "shifted" modules in the setting of semigroups more general than those arising from the morphisms of a category.

**Lemma 2.3.** Let \( N \in R_{-gr} \), \( f, g \in M(C) \), and assume that \( f \circ g \) is defined. Then

\[
N(f \circ g) = (N(f))(g).
\]

**Proof.** Let \( k \in M(C) \). Then

\[
((N(f))(g))_k = \sum_h \{N(f)_h \mid h \circ g \text{ is defined and } h \circ g = k\}
\]

\[
= \sum_h \sum_t \{N_t \mid t \circ f \text{ is defined and } t \circ f = h, \text{ and } h \circ g \text{ is defined and } h \circ g = k\}
\]

\[
= \sum_t \{N_t \mid t \circ (f \circ g) \text{ is defined and } t \circ (f \circ g) = k\}
\]

\[
= (N(f \circ g))_k. \quad \Box
\]

We denote by \( F : R_{-gr} \to R_{-Mod} \) the forgetful functor; specifically, given \( N \in R_{-gr} \), \( F(N) = N \) when regarded as a left \( R \)-module.

**Corollary 2.4.** Let \( N \) be a graded \( R \)-module, and let \( f \in M(C) \) with \( D = D(f) \).

(a) \( F(N(f)) = F(N(1_D)) \).

(b) \( N(1_D)(f) = N(f) \).

(c) For every \( B \in \text{Ob}(C) \), \( N(1_B) \) is a graded submodule of \( N \) and

\[
N = \bigoplus_{B \in \text{Ob}(C)} N(1_B) \quad \text{in } R_{-gr}.
\]

**Proof.** Part (a) follows immediately by definition, and (b) is clear from the above lemma. Part (c) follows by noting that for \( B \in \text{Ob}(C) \) and \( h \in M(C) \), \( N(1_B)_h = N_h \) if \( B = C(h) \) and is zero otherwise. \( \Box \)
Proposition 2.5. The collection \{R(f) \mid f \in M(C)\} is a family of generators for \(R-\text{gr}\). In particular, \(R-\text{gr}\) is a Grothendieck category.

Proof. Let \(A, A' : N \to N'\) be morphisms in \(R-\text{gr}\) and assume that \(A \neq A'\). Then there exists \(f \in M(C)\) and \(0 \neq x \in N_f\) such that \((x)A \neq (x)A'\). As \(N = RN\) we may assume without loss of generality that \(x = ry\) for suitable \(r \in R_g, y \in N_h\), where \(g, h \in M(C)\), \(g \circ h\) is defined, and \(g \circ h = f\). By definition we in fact have \(r \in (R(h))_f\). It is now easy to check that the mapping \(a \mapsto ay\) defines a graded morphism \(A : R(h) \to N\) having \((r)x = ry = x\). Thus \((r)xA = (x)A \neq (x)A' = (r)xA'\), which yields the desired result. \(\square\)

Definition 2.6. Let \(R\) be a ring graded by the category \(C\). We call \(R\) locally unital in case
(i) The ring \(R_{1_B}\) is unital for every \(B \in \text{Ob}(C)\) (we denote the identity of \(R_{1_B}\) by \(1_B\)), and
(ii) for each \(g \in \text{Mor}(B,B')\) and \(r \in R_g\), \(1_Br = r = r1_{B'}\).

Examples of locally unital rings graded by a category abound. For instance, it is easy to show that each of the examples described subsequent to Definition 2.1 is locally unital. In case \(C\) is a group, then a straightforward check shows that the ring \(R\) graded by \(C\) is locally unital exactly when it is unital. More generally, a locally unital ring \(R\) graded by the category \(C\) is unital if and only if \(\text{Ob}(C)\) is finite; in this case, \(1 = \sum_{B \in \text{Ob}(C)} 1_B\).

Proposition 2.7. If \(R\) is a locally unital ring graded by the category \(C\), then the collection \(\{R(f) \mid f \in M(C)\}\) is a family of finitely generated projective generators for \(R-\text{gr}\).

Proof. It is straightforward to show that \(R(1_B) = R1_B\) as left \(R\)-modules for each object \(B\) of \(C\). Now consider an exact diagram

\[
\begin{array}{ccc}
R(f) & \to & N \\
\downarrow^x & & \\
M & \to & 0
\end{array}
\]

in \(R-\text{gr}\), and let \(B = D(f)\). Then, by Corollary 2.4, \(F(R(f)) = F(R(1_B)) = R1_B\); moreover, \(1_R \in (R(f))_f\). Hence, \((1_R)x \in N_f\). Since, \(\beta\) is graded and surjective we have \((1_R)x = (x_f)\beta\) for some \(x_f \in M_f\). Then \(y : R(f) \to M\) defined by \((r1_B)\gamma = r1_{BG}\) is a morphism in \(R-\text{gr}\) which makes the diagram commute. Note that \(R(f)\) is finitely generated in \(R-\text{gr}\) since, as a left \(R\)-module, it is just \(R1_B\). The result now follows from Proposition 2.5. \(\square\)
With the above proposition in hand, we now apply Theorem 1.9 of Section 1 to locally unital rings graded by categories to get:

**Theorem 2.8.** Let $C$ be any category such that the collection of morphisms $M(C)$ is a set. Let $R = \bigoplus_{f \in M(C)} R_f$ be a locally unital ring graded by $C$ and let $R_{-\text{gr}}$ denote the category of graded left $R$-modules. Let $A$, $\hat{U}$, and $I$ denote the rings and modules described in Section 1 corresponding to the category $R_{-\text{gr}}$ and the family of generators $\{R(f) \mid f \in M(C)\}$. Then the functor

$$H^*_{R_{-\text{gr}}} (\hat{U}, -): R_{-\text{gr}} \to \text{Gen}(A)$$

is a category equivalence. In particular,

$$H^1_{R_{-\text{gr}}} (\hat{U}, -): R_{-\text{gr}} \to \text{Gen}(I)$$

is a category equivalence.

We will conclude this article by describing generalizations to rings graded by categories of the various smash product constructions for group-graded rings which were discussed in the introduction. The goal here is to give a concrete, matricial description of the rings which "realize" $R_{-\text{gr}}$ for a ring $R$ graded by a category.

For morphisms $h, k$ in $C$ we set

$$C^t_h = \{ f \in M(C) \mid fk \text{ is defined, and } fk = h \} .$$

**Definition 2.9.** We denote by $RAC$ the subring of the ring $FCM_{M(C)}(R)$ of all $M(C)$-square column finite matrices over $R$ defined by

$$RAC = \left\{ A \in FCM_{M(C)}(R) \mid A_{f,g} \in \bigoplus_t R_t \text{ for } t \in C^t_f \text{ and } A_{f,g} = 0 \text{ if } C^t_f = \phi \right\} .$$

The connection between $RAC$ and the rings described in Section 1 will follow from

**Lemma 2.10.** Let $R$ be a locally unital ring graded by the category $C$. Given $f, g \in M(C)$ we have

$$\text{Hom}_{R_{-\text{gr}}}(R(f), R(g)) \cong \bigoplus_{f = tg} R_t$$

via right multiplication.

**Proof.** Let $B = D(f)$. By Corollary 2.4 and the proof of Proposition 2.5 we have $R(f) = (R_{1B})(f) = (R_1B)(f)$. Therefore, every $\alpha \in \text{Hom}_{R_{-\text{gr}}}(R(f), R(g))$ is completely determined by $(1_B)\alpha$. As $1_B \in ((R_1B)(f))_f$, we must have $(1_B)\alpha \in R(g)_f = \bigoplus_{f = tg} R_t$ by the definition of graded morphisms. On the other hand, if $a \in \bigoplus_{f = tg} R_t$, then $a = 1_B a$. Thus, the map $\alpha: R(f) \to R(g)$ defined via $r1_B \mapsto ra$ is well defined, and is a graded morphism. □
Using Lemma 2.10 along with Definitions 1.3 and 2.2, it is clear that the ring \( R \Delta C \) is precisely the ring \( H(U) \) in the particular setting of graded rings and modules. Similarly, the ideal \( I(U) \) of \( H(U) \) corresponds to the ideal \( K \) of \( R \Delta C \) consisting of those matrices which have at most finitely many nonzero entries. Moreover, given \( N = \bigoplus_{f \in M(C)} N_f \) in \( R-\text{gr} \), \( H^i_{R-\text{gr}}(U, N) \) can be regarded as the abelian group \( N \) endowed with the left \( A \)-module structure defined via

\[
An = \sum_{f \in M(C)} \left( \sum_{g \in M(C)} A_{f,g} n_g \right)
\]

for \( A = (A_{f,g}) \in A \) and \( n = \sum_{g \in M(C)} n_g \in N \). Thus using the results of Section 1, along with Theorem 2.8, we get the following matricial description of the category \( R-\text{gr} \):

**Theorem 2.11.** Let \( R \) be a locally unital ring graded by the category \( C \). Let \( K \) denote the ideal of \( R \Delta C \) consisting of those matrices which have at most finitely many nonzero entries. Let \( A \) be any subring of \( R \Delta C \) which contains \( K \). Then the category \( R-\text{gr} \) of \( C \)-graded left \( R \)-modules is equivalent to the category \( \text{Gen}(A K) \) of left \( A \)-modules generated by \( K \).

The above theorem immediately yields each of the realization results offered in the works of Cohen–Montgomery, Beattie, and Albu–Năstăsescu which were mentioned in the introduction.

When \( C \) is finite, then clearly the rings \( R \Delta C \), \( A \), and \( K \) mentioned in the above result coincide. In this situation we denote these rings by \( R \# C \); we point out below that when \( C \) is finite this notation is consistent with the smash product notation used by other authors. Using the fact that for any unital ring \( T \) we have \( \text{Gen}(T T) = T-\text{Mod} \), we immediately get the following corollary to the above Theorem in this situation.

**Corollary 2.12.** Let \( R \) be a locally unital ring graded by the finite category \( C \). Then the category \( R-\text{gr} \) of \( C \)-graded left \( R \)-modules is equivalent to the category \( R \# C-\text{Mod} \) of left \( R \# C \)-modules.

We conclude this article with a few observations. First, if the category \( C \) is right-cancellative (i.e., for all \( f, f', g \in M(C) \) with \( f \circ g \) and \( f' \circ g \) defined, if \( f \circ g = f' \circ g \), then \( f = f' \)), then the ring \( R \Delta C \) may be described as

\[
R \Delta C = \{ A \in FCM_{M(C)}(R) \mid A_{f,g} R_{f,g} \in R, \text{ if } tg \text{ is defined, and is } 0 \text{ elsewhere} \};
\]

this is completely analogous to the corresponding construction in the group-graded case.

Secondly, we point out some of the relationships between our construction and Quinn's smash product ring. Let \( C \) be any category. Then the ring \( R \) embeds in the ring \( R \Delta C \) as follows. Let \( \rho : R \to R \Delta C \) be the map which takes each \( r = \sum_{f \in M(C)} r_f \).
to $\tilde{r}$, where $\tilde{r}_{h,k} = \sum_{f \in C} r_f$ if $C^k \neq \phi$ and $\tilde{r}_{h,k} = 0$ otherwise. We see that $\tilde{r}$ always has finite columns. (In general, however, $\tilde{r}$ need not have finite rows; in fact, for a fixed $r_f \neq 0$, it is possible to have $h \in M(C)$ such that $fk = h$ for infinitely many $k$.) It is routine to prove that $\rho$ is an injective ring homomorphism, with $\rho(1) = 1$ whenever $1 \in R$ and $R$ is locally unital. (We note that if $C$ happens to be right cancellative and $R$ is unital, then one can show that $R$ is in fact locally unital.) Now, following Quinn, one may consider the ring $R\#C$; this is defined to be the ring spanned by $\tilde{R}$ and $\{p_f\}_{f \in M(C)}$, where $p_f$ is the matrix having $(f,f)$-component equal to $1_{D(f)}$ and is 0 elsewhere. For given $r_i \in R_t$, $\tilde{r} \cdot p_f = 0$ unless $tf$ is defined. In this case, $\tilde{r} \cdot p_f$ is the matrix having $(tf,f)$-entry equal to $r_i$ and is 0 elsewhere. From this we conclude that $\sum_{f \in M(C)} \tilde{R}p_f = K$, so that $R\#C$ can be regarded as the smallest subring of $RAC$ which contains both $K$ and $\tilde{R}$. Note that, since the $\{p_f\}$ are orthogonal, $K = \bigoplus_{f \in M(C)} \tilde{R}p_f$. Moreover, a routine check shows that $R1_{D(f)} = \tilde{R}p_f$ for each $f \in M(C)$, so that $K$ is a projective left $\tilde{R}$-module.

However, in contrast to the group-graded situation, we need not in general have $p_f\tilde{r}_i \in K$, as $(p_f\tilde{r}_i)_{f,k} = r_i$ for all $k$ having $tk = f$. When $C$ is left-cancellative (i.e., for all $f,g,g' \in M(C)$ with $f \circ g$ and $f \circ g'$ defined, if $f \circ g = f \circ g'$ then $g = g'$) we do in fact get $p_f\tilde{r}_i \in K$ for all $f,t \in M(C)$, which yields (as in the group-graded case) $\tilde{R}\#C = \tilde{R} + K$. Nevertheless, if $M(C)$ is infinite, we cannot mimic the group case to conclude that $\tilde{R} \cap K = \{0\}$. Specifically, given $0 \neq r_i \in R_t$, it can happen that $tg$ is defined only for finitely many $g$. In further contrast to the group case, we cannot conclude in general that $K$ equals $\sum_{f \in M(C)} p_f\tilde{R}$ unless $C$ is left cancellative (in the left-cancellative case we get $p_f\tilde{r}_i = \tilde{r}_i p_f$ whenever $tf$ is defined). Additionally, we note that even if $C$ is finite and both left and right cancellative we need not have that $K$ is a projective right $\tilde{R}$-module (details are provided in [2]).

We briefly note in conclusion that our ring $RAC$ is not the ring $RV\gamma C$ described in [4]; the ring $RV\gamma C$ contains row-finite entries, while our ring $R\Delta C$ contains column-finite entries. Although this distinction between $RV\gamma C$ and $R\Delta C$ may on the surface seem superficial, we will describe in a future article the inherent differences between them.

Appendix

A semigroup $S$ (possibly with zero) is said to have local identities in case $S$ contains a subset $E$ of orthogonal idempotents, having the property that for any nonzero $f \in S$ there exist unique elements $e,e' \in E$ with $efe' = f$. If $C$ is a category whose morphisms $M(C)$ form a set, then $M(C)$ may clearly be viewed as a semigroup having local identities. In the article “Realization theorems for categories of graded modules over semigroup-graded rings” (Comm. Algebra 22, 1994, 5343–5388), the authors and del Rio have proved a result analogous to Proposition 2.7 for all semigroups having local identities and finite subset $E$. (A redefinition of modules of the form $R(f)$ is required.) This in turn implies that all the results contained in the current article subsequent to Proposition 2.7 are in fact valid in this setting as well.
References