Integrals, quantum Galois extensions and the affineness criterion for quantum Yetter-Drinfel’d modules

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Abstract
In this paper we shall generalize the notion of integral on a Hopf algebra introduced by Sweedler, by defining the more general concept of integral of a threetuple \((H, A, C)\), where \(H\) is a Hopf algebra coacting on an algebra \(A\) and acting on a coalgebra \(C\). We prove that there exists a total integral \(\gamma : C \to \text{Hom}(C, A)\) of \((H, A, C)\) if and only if any representation of \((H, A, C)\) is injective in a functorial way, as a corepresentation of \(C\). In particular, the quantum integrals associated to Yetter-Drinfel’d modules are defined. Let now \(A\) be an \(H\)-bicomodule algebra, \(\mathcal{H}YD_A\) the category of quantum Yetter-Drinfel’d modules and \(B = \{a \in A \mid \sum S^{-1}(a_{<1>}a_{<-1>} \otimes a_{<0>} = 1_H \otimes a\}\), the subalgebra of coinvariants of the Verma structure \(A \in \mathcal{H}YD_A\). We shall prove the following affineness criterion: if there exists \(\gamma : H \to \text{Hom}(H, A)\) a total quantum integral and the canonical map \(\beta : A \otimes_B A \to H \otimes A, \beta(a \otimes_B b) = \sum S^{-1}(b_{<1>}b_{<-1>} \otimes ab_{<0>}\) is surjective (i.e. \(A/B\) is a quantum homogeneous space), then the induction functor \(- \otimes_B : \mathcal{M}_B \to \mathcal{H}YD_A\) is an equivalence of categories. The affineness criteria proven by Cline, Parshall and Scott, and independently by Oberst (for affine algebraic groups schemes), Schneider (in the noncommutative case), are recovered as special cases.

0 Introduction

The integrals for Hopf algebras were introduced in two fundamental papers: by Larson and Sweedler in [32] for the finite case, and by Sweedler in [48] for the infinite case. Initially introduced in order to generalize the Haar measure on a compact group, the integrals have proven to be a powerful instrument in the classic theory of Hopf algebras, beginning with representation theory and ending with the classification theory for finite dimensional Hopf algebras. Recently, arising from an idea of Drinfel’d, the integrals were introduced for Hopf algebras in various braided categories: abelian and rigid ([34]) or rigid with split idempotents ([2]). At this level (mutatis mutandis, the definition is fundamentally the one given by Sweedler), integrals have proven to be essential instruments in constructing invariants of surgically presented 3-manifolds or 3-dimensional topological quantum field theories ([27], [30], [51]).

∗This paper was written while the first author was a member of G.N.S.A.G.A. with partial financial support from M.U.R.S.T. and the second author was a visiting professor at the University of Ferrara, supported by C.N.R. and C.N.C.S.I.S.
In the first part of this paper we shall introduce the more general concept of integral associated to a threetuple \((H, A, C)\) called Doi-Koppinen datum, consisting of a Hopf algebra \(H\) which coacts on an algebra \(A\) and acts on a coalgebra \(C\). As a major application, the quantum integrals associated to Yetter-Drinfel’d modules \(H \mathcal{YD}_H\), are introduced. The transition from the classic integrals of Sweedler, which are elements \(\varphi \in H^*\) invariant to convolution (or equivalently \(H\)-colinear maps \(\varphi : H \to k\)), to the quantum integrals, which are maps \(\gamma : H \to \text{End}(H)\) satisfying the condition

\[
\sum g(1) \otimes \gamma(g(2))(h) = \sum S^{-1}\{(\gamma(g)(h_{(1)}))_{(2)}\} h_{(2)}\{\gamma(g)(h_{(1)})\}_{(1)} \otimes \{\gamma(g)(h_{(1)})\}_{(2)}
\]

for all \(g, h \in H\), is a long way which needs to be explained.

First of all, the integrals on a Hopf algebra \(H\) and the more general ones introduced by Doi ([19]) for an \(H\)-comodule algebra \(A\), have strong ties to \(\mathcal{M}^H\), the corepresentations of \(H\), and to the representations of the pair \((H, A)\), being the category of relative Hopf modules \(\mathcal{M}^H_A\). The starting point for this paper is presented in Section 1.2: the existence of an integral in the sense of Doi (classic, if we consider \(A = k\)) is the necessary and sufficient criterion for the existence of a natural transformation between two functors linking \(\mathcal{M}^H_A\) to \(\mathcal{M}^H\) (see Theorem 1.2).

This categorical point of view towards integrals will allow us to correctly define the integrals associated to a Doi-Koppinen datum \((H, A, C)\): an object of it (also called a Doi-Koppinen module) is a \(k\)-module with an \(A\)-action and a compatible \(C\)-coaction. In view of the above, an integral of \((H, A, C)\) will be a map, the existence of which is equivalent with the existence of a natural transformation between the functors \(F_A \circ (C \otimes \bullet) \circ F^C\) and \(F_A \circ 1_{C\mathcal{M}(H)_A} : C\mathcal{M}(H)_A \to C\mathcal{M}\), where \(F_A : C\mathcal{M}(H)_A \to C\mathcal{M}\) and \(F^C : C\mathcal{M}(H)_A \to \mathcal{M}_A\) are the corresponding forgetful functors.

The definition of the integral of \((H, A, C)\), i.e. a map \(\gamma : C \to \text{Hom}(C, A)\) satisfying the equation (29), is given in Definition 2.1, and its characterization in Theorem 2.6, which can be interpreted as follows: there exists a total integral \(\gamma : C \to \text{Hom}(C, A)\) of \((H, A, C)\) if and only if any Doi-Koppinen module is relative injective (injective, if we work over a field) in a functorial way, as a left \(C\)-comodule. A key result is Theorem 2.9: if there exists a total integral \(\gamma : C \to \text{Hom}(C, A)\) of \((H, A, C)\), then \(C \otimes A\) is a generator in the category \(C\mathcal{M}(H)_A\). As explained in previous publications ([10], [12]), \(C\mathcal{M}(H)_A\) unifies modules, comodules, Sweedler’s Hopf modules, relative Hopf modules, graded modules, Long dimodules and Yetter-Drinfel’d modules. In particular, by applying the above results we obtain the definition and the characterization theorem for quantum integrals, being the integrals which correspond to the Yetter-Drinfel’d modules \(H \mathcal{YD}_H\): If there exists a total quantum integral \(\gamma : H \to \text{End}(H)\), then any Yetter-Drinfel’d module \(M \in H \mathcal{YD}_H\) is relative injective as an \(H\)-comodule. In particular, if \(H\) is finite dimensional over a field \(k\), there exists a total quantum integral \(\gamma : H \to \text{End}(H)\) if and only if any representation of the Drinfel’d double \(D(H)\) is injective in a functorial way, as an \(H^*\)-module.

In the second part of the paper we introduce the notion of quantum Galois extensions and we prove a criterion for affineness in a quantum version. Let us explain the terminology and justify the usage of the term ”quantum”. We shall begin by recalling the following powerful theorem given by Schneider ([46]) \(^1\) (presenting an equivalent right-left version of it):

**Theorem 0.1** Let \(H\) be a Hopf algebra with a bijective antipode over a field \(k\), \(A\) a left \(H\)-comodule algebra and \(B = A^{co(H)}\) its subalgebra of coinvariants. The following statements are equivalent

\(^1\)For \(H\) and \(A\) commutative, this result was proved before by Doi in [19, Theorem 3.2].
1. (a) there exists \( \varphi : H \rightarrow A \) a total integral; 
   (b) the canonical map \( \text{can} : A \otimes_B A \rightarrow H \otimes A, \text{can}(a \otimes_B b) = \sum b_{<1>} \otimes ab_{<0>} \), is surjective;

2. the induction functor \( - \otimes_B A : \mathcal{M}_B \rightarrow \mathcal{M}_A \) is an equivalence of categories;

3. (a) \( A \) is faithfully flat as a left \( B \)-module; 
   (b) \( A/B \) is an \( H \)-Galois extension, i.e. \( \text{can} \) is bijective.

A few comments on this result: 
3) \( \Rightarrow \) 1) was proven for the case when \( k \) is a field, \(^2\) based on a result of Takeuchi [50]: over a field, an \( H \)-comodule is injective if and only if it is coflat. All the other implications hold true over a commutative ring, if we additionally assume that \( H \) is projective over \( k \). The equivalence 2) \( \Leftrightarrow \) 3) is a general imprimitivity theorem and its proof is standard from the point of view of category theory: a pair of adjoint functors (as \( - \otimes_B A \) and \( (-)^{\text{co}(H)} \)) are given an equivalence of categories if one of them is faithfully exact (or both of them are exact) and the adjunction maps in the key objects of categories (\( B \) in \( \mathcal{M}_B \) and \( H \otimes A \) in \( \mathcal{M}_A \)) are bijective. \(^3\) The main part of the theorem is 1) \( \Rightarrow \) 2), which is the non-commutative version of the affineness criterion for affine algebraic groups schemes given before by Cline, Parshall and Scott [16] and independently by Oberst [42].

Our intention is to “quantize” this result. The first step to be taken is to quantize the category \( \mathcal{H} \mathcal{M}_A \) in a coherent way. In Hopf algebras theory, quantization means (roughly speaking) a deformation of the enveloping algebra of a semisimple Lie algebra \( \ell \) using a parameter \( q \), in order to obtain a noncommutative noncocommutative Hopf algebra (a quantum group) \( U_q(\ell) \) ([22], [25]). The results obtained for the new object \( U_q(\ell) \) in the framework of representation theory ([33], [45]) will generalize the results from the classic case \( U(\ell) \). In order to quantize the category \( \mathcal{H} \mathcal{M}_A \), the part of the parameter \( q \) mentioned above will be played by a new coaction (i.e. a family of parameters, possible infinite, if \( k \) is a field), this time to the right, \( \rho^r : A \rightarrow A \otimes H \). Thus \( A \) will be a \( H \)-bicocomodule algebra and the category of representations will be denoted by \( \mathcal{H} \mathcal{Y}D_A \), the category of quantum Yetter-Drinfel’d modules introduced in [12]. If \( A = H \) and \( \rho^l = \rho^r = \Delta \) then \( \mathcal{H} \mathcal{Y}D_H \) is the category of Yetter-Drinfel’d (or crossed) modules introduced in [22], [52]; if \( H \) is finite dimensional then \( \mathcal{H} \mathcal{Y}D_H \cong \mathcal{M}_{D(H)} \), where \( D(H) \) is the Drinfel’d double of \( H \) ([35]). The category \( \mathcal{H} \mathcal{Y}D_H \) plays an important role in the quantum Yang-Baxter equation, low dimensional topology or knot theory ([26], [36]). In [23], the category of Yetter-Drinfel’d modules\(^4\) was used as the fundamental tool in the construction of the dequantization functor \( DQ \), which is an equivalence of categories from the category of quantized universal enveloping algebras to the category of Lie bialgebras over \( k[[h]] \).

On the other hand, if \( \rho_r : A \rightarrow A \otimes H \) is the trivial coaction, that is \( \rho^r(a) = a \otimes 1_H \), then \( \mathcal{H} \mathcal{Y}D_A = \mathcal{H} \mathcal{M}_A \), the category of classical relative Hopf modules. Hence, \( \mathcal{H} \mathcal{Y}D_A \) is a category containing the category of Yetter-Drinfel’d modules \( \mathcal{H} \mathcal{Y}D_H \) as a particular case and, on the other hand, \( \mathcal{H} \mathcal{M}_A \) is obtained from \( \mathcal{H} \mathcal{Y}D_A \) by trivializing the right coaction of \( H \) on \( A \). We can therefore view the category \( \mathcal{H} \mathcal{Y}D_A \) as a quantization of the category of relative Hopf modules \( \mathcal{H} \mathcal{M}_A \). For this reason, throughout this paper we shall call the objects of \( \mathcal{H} \mathcal{Y}D_H \) Yetter-Drinfel’d modules, while the objects of the more general category \( \mathcal{H} \mathcal{Y}D_A \) shall be called quantum Yetter-Drinfel’d modules.

Now we can extend Theorem 0.1 from relative Hopf modules to quantum Yetter-Drinfel’d modules: this is what we shall do in Section 3. With the exception of 3) \( \Rightarrow \) 1) (which remains an open

\(^2\)Recently, in [38], it was proven using other techniques that this implication holds for commutative QF-rings.

\(^3\)For generalizations of this equivalence we refer to [5, Theorem 5.6], [6, Theorem 3.10] or [14, Theorem 2.8].

\(^4\)The authors consider the left-left equivalent version of \( \mathcal{H} \mathcal{Y}D_H \) and the object of it are called \( H \)-dimodules.
problem), all the other implications maintain their validity for the category $^H \mathcal{YD}_A$. Let

$$B = \{ a \in A \mid \sum S^{-1}(a_{<1>}) a_{<-1>} \otimes a_{<0>} = 1_H \otimes a \},$$

be the subalgebra of quantum coinvariants, which are the coinvariants of the Verma structure $A \in ^H \mathcal{YD}_A$. Proposition 3.11 shows that the induction functor $- \otimes_B A : \mathcal{M}_B \to ^H \mathcal{YD}_A$ is an equivalence of categories if and only if $A/B$ is a faithfully flat quantum Galois extension, i.e. the canonical map $\beta : A \otimes_B A \to H \otimes A$, $\beta(a \otimes_B b) = \sum S^{-1}(b_{<1>}) b_{<-1>} \otimes ab_{<0>}$ is bijective. Theorem 3.15 proves the quantum affineness criterion: if there exists $\gamma : H \to \text{Hom}(H, A)$ a total quantum integral and the canonical map $\beta : A \otimes_B A \to H \otimes A$ is surjective, then the induction functor $- \otimes_B A : \mathcal{M}_B \to ^H \mathcal{YD}_A$ is an equivalence of categories.

1 Preliminary results

Throughout this paper, $k$ will be a commutative ring with unit. Unless specified otherwise, all modules, algebras, coalgebras, bialgebras, tensor products and homomorphisms are over $k$. For a $k$-algebra $A$, $\mathcal{M}_A$ (resp. $\mathcal{M}^C$) will be the category of right (resp. left) $A$-modules and $A$-linear maps. $H$ will be a Hopf algebra over $k$, and we will use Sweedler’s sigma-notation extensively. For example, if $(C, \Delta)$ is a coalgebra, then for all $c \in C$ we write

$$\Delta(c) = \sum c_{(1)} \otimes c_{(2)} \in C \otimes C, \quad (\Delta \otimes \text{Id})\Delta(c) = (\text{Id} \otimes \Delta)\Delta(c) = \sum c_{(1)} \otimes c_{(2)} \otimes c_{(3)}.$$ 

If $(M, \rho_M)$ is a left $C$-comodule, then we write

$$\rho_M(m) = \sum m_{<-1>} \otimes m_{<0>} \in C \otimes M,$$ 

and

$$(\Delta \otimes \text{Id})\rho_M(m) = (\text{Id} \otimes \rho_M)\rho_M(m) = \sum m_{<-2>} \otimes m_{<-1>} \otimes m_{<0>} \in C \otimes C \otimes M$$

for $m \in M$. $\mathcal{C} M$ (resp. $\mathcal{M}^C$) will be the category of left (resp. right) $C$-comodules and $C$-colinear maps. If $M$ is a left $C$-comodule then $C \otimes M$ is also a left $C$-comodule via $\rho_{C \otimes M} := \Delta \otimes \text{Id}_M$ and $\rho_M : M \to C \otimes M$ is a left $C$-colinear map. A left $C$-comodule $M$ is called relative injective if for any $k$-split monomorphism $i : U \to V$ in $\mathcal{C} M$ and for any $C$-colinear map $f : U \to M$, there exists a $C$-colinear map $g : V \to M$ such that $g \circ i = f$. This is equivalent ([19]) to the fact that $\rho_M : M \to C \otimes M$ splits in $\mathcal{C} M$, i.e. there exists a $C$-colinear map $\lambda_M : C \otimes M \to M$ such that $\lambda_M \circ \rho_M = \text{Id}$. Of course, if $k$ is a field, $M$ is relative injective if and only if it is an injective object in $\mathcal{C} M$.

The dual $C^* = \text{Hom}(C, k)$ of a $k$-coalgebra $C$ is a $k$-algebra. The multiplication on $C^*$ is given by the convolution

$$\langle f \ast g, c \rangle = \sum \langle f, c_{(1)} \rangle \langle g, c_{(2)} \rangle,$$

for all $f, g \in C^*$ and $c \in C$. $C$ is a $C^*$-bimodule: the left and right action are given by the formulas

$$c^* \ast c = \sum \langle c^*, c_{(2)} \rangle c_{(1)} \quad \text{and} \quad c \ast c^* = \sum \langle c^*, c_{(1)} \rangle c_{(2)}$$

for $c^* \in C^*$ and $c \in C$. This also holds for $C$-comodules: for example, if $(M, \rho_M)$ is a left $C$-comodule, then it becomes a right $C^*$-module by

$$m \cdot c^* = \sum \langle c^*, m_{<-1>} \rangle m_{<0>}.$$
for all \( m \in M \) and \( c^* \in C^* \).

An algebra \( A \) that is also a left \( H \)-comodule is called a left \( H \)-comodule algebra if the comodule structure map \( \rho_A : A \to H \otimes A \) is an algebra map. This means that

\[
\rho_A(ab) = \sum a_{<-1>}b_{<-1>} \otimes a_{<0>}b_{<0>} \text{ and } \rho_A(1_A) = 1_H \otimes 1_A
\]

for all \( a, b \in A \). This is equivalent to the fact that \( A \) is an algebra in the monoidal category \( H \mathcal{M} \) of left \( H \)-comodules.

Similarly, a coalgebra that is also a right \( H \)-module is called a right \( H \)-module coalgebra if \( C \) is a coalgebra in the monoidal category \( \mathcal{M}_H \) of right \( H \)-modules, or equivalent

\[
\Delta_C(c \cdot h) = \sum c_{(1)} \cdot h_{(1)} \otimes c_{(2)} \cdot h_{(2)} \text{ and } \varepsilon_C(c \cdot h) = \varepsilon_C(c)\varepsilon_H(h),
\]

for all \( c \in C, h \in H \).

### 1.1 Doi-Koppinen modules: functors and structures

Let \( H \) be a Hopf algebra, \( A \) a left \( H \)-comodule algebra and \( C \) a right \( H \)-module coalgebra. The threepuple \( \mathcal{O} = (H, A, C) \) is called a Doi-Koppinen datum. In order to study these general objects, we have to define their representations: a representation of \( 1 \mathcal{H}_0 \)-comodule \( \rho \)-colinear and can be viewed as a natural transformation between the functors \( 1 \mathcal{H}_0 \)-comodule algebra if the comodule \( C, \rho \)-colinear and can be viewed as a natural transformation between the functors \( 1 \mathcal{H}_0 \)-comodule algebra if the comodule structures map on \( \mathcal{C} \mathcal{M}(H)_A \) will be the abelian category of right-left Doi-Koppinen modules and \( A \)-linear, \( C \)-colinear maps, as it was introduced by Y. Doi in [20] and independently by M. Koppinen in [28]. Let \( F^C : C \mathcal{M}(H)_A \to \mathcal{M}_A \) be the forgetful functor which forgets the \( C \)-coaction and

\[
\text{its right adjoint, where the structure maps on } C \otimes M \text{ are given by}
\]

\[
(c \otimes m) \cdot a = \sum c \cdot a_{<-1>} \otimes ma_{<0>}
\]

\[
\rho_{C \otimes M}(c \otimes m) = \sum c_{(1)} \otimes c_{(2)} \otimes m
\]

for any \( c \in C, a \in A \) and \( m \in M \). The unit of the adjoint pair \( (F^C, C \otimes \bullet) \) is precisely ([10], [39])

\[
\rho : 1_{C \mathcal{M}(H)_A} \to (C \otimes \bullet) \circ F^C,
\]

the \( C \)-coaction \( \rho_M : M \to C \otimes M \) on any Doi-Koppinen module \( M \); therefore \( \rho_M \) is \( A \)-linear and \( C \)-colinear and can be viewed as a natural transformation between the functors \( 1_{C \mathcal{M}(H)_A} \) and \( (C \otimes \bullet) \circ F^C \). \( A \) is a right \( A \)-module, so \( C \otimes A \) is a Doi-Koppinen module via:

\[
(c \otimes b)a = \sum ca_{<-1>} \otimes ba_{<0>}
\]

\[
\rho^L_{C \otimes A}(c \otimes b) = \sum c_{(1)} \otimes c_{(2)} \otimes b
\]

Furthermore, \( C \otimes A \) is also a right \( C \)-comodule via

\[
\rho_{C \otimes A}^r : C \otimes A \to C \otimes A \otimes C, \quad \rho_{C \otimes A}^r(c \otimes a) = \sum c_{(1)} \otimes a_{<0>} \otimes c_{(2)}S(a_{<-1>})
\]
for all \(a \in A, c \in C\). \(C \otimes C \otimes A = \left((C \otimes \bullet) \circ F^C\right)(C \otimes A)\) is an object in \(C \mathcal{M}(H)_A\) and also has a right \(C\)-comodule structure via:

\[
(c \otimes d \otimes b)a = \sum c_{-2}a \otimes d_{-1} \otimes ba_{0}\tag{8}
\]

\[
\rho^l_{C \otimes C \otimes A}(c \otimes d \otimes b) = \sum c_{(1)} \otimes c_{(2)} \otimes d \otimes b\tag{9}
\]

\[
\rho^r_{C \otimes C \otimes A}(c \otimes d \otimes b) = c \otimes \rho^l_{C \otimes A}(d \otimes b) = \sum c \otimes d_{(1)} \otimes b_{0} \otimes d_{(2)}S(b_{-1})\tag{10}
\]

Now, let \(F_A : C \mathcal{M}(H)_A \to C \mathcal{M}\) be the other forgetful functor, which forgets the \(A\)-action and

\[
\bullet \otimes A : C \mathcal{M} \to C \mathcal{M}(H)_A, \quad N \to N \otimes A
\]

its left adjoint, where for \(N \in C \mathcal{M}, N \otimes A \in C \mathcal{M}(H)_A\) via the structures

\[
(n \otimes a) \cdot b = n \otimes ab\tag{11}
\]

\[
\rho_{N \otimes A}(n \otimes a) = \sum n_{-1}a_{-1} \otimes n_{0} \otimes a_{0}\tag{12}
\]

for any \(a, b \in A\) and \(n \in N\). \(C\) is a left \(C\)-comodule via \(\Delta\); hence \(C \otimes A\) can be also viewed as a Doi-Koppinen module via

\[
(c \otimes b) \cdot a = c \otimes ba\tag{13}
\]

\[
\rho^l_{C \otimes A}(c \otimes b) = \sum c_{(1)} \cdot b_{-1} \otimes c_{(2)} \otimes b_{0}\tag{14}
\]

for all \(c \in C, a, b \in A\). These two types of Doi-Koppinen module structures on \(C \otimes A\), coming from (13) and (14) or from (5) and (6), are isomorphic: more precisely, the map

\[
u : C \otimes A \to C \otimes A, \quad u(c \otimes a) = \sum ca_{-1} \otimes a_{0}\tag{15}\]

is an isomorphism of Doi-Koppinen modules ([14]) with an inverse given by

\[
u^{-1} : C \otimes A \to C \otimes A, \quad \nu^{-1}(c \otimes a) = \sum cS(a_{<1>}) \otimes a_{0}\]

The algebra \(C^*\) is a left \(H\)-module algebra; the \(H\)-action is given by the formula

\[
\langle h \cdot c^*, c \rangle = \langle c^*, c \cdot h \rangle
\]

for all \(h \in H, c \in C\) and \(c^* \in C^*\). The smash product \(A \# C^*\) is equal to \(A \otimes C^*\) as a \(k\)-module, with the multiplication defined by

\[
(a \# c^*)(b \# d^*) = \sum a_{0}b \# c^* \cdot (a_{<1>} \cdot d^*),\tag{16}
\]

for all \(a, b \in A, c^*, d^* \in D^*\). Recall that we have a natural functor \(P : C \mathcal{M}(H)_A \to \mathcal{M}_{A \# C^*}\), sending a Doi-Koppinen module \(M\) to itself, with the right \(A \# C^*\)-action given by

\[
m \cdot (a \# c^*) = \sum \langle c^*, m_{<1>} \rangle m_{<0>}a\tag{17}
\]

for any \(m \in M, a \in A\) and \(c^* \in C^*\). \(P\) is an equivalence of categories if \(C\) is finitely generated and projective as a \(k\)-module ([20]).

As in [10], the following right \(C^*\)-module on \(\text{Hom}(C, A)\) will play a key role

\[
(f \cdot c^*)(c) = \sum f(c_{(1)})_{<0>} \langle c^*, c_{(2)} \cdot f(c_{(1)})_{<1>} \rangle\tag{18}
\]
for any \( f \in \text{Hom}(C, A) \), \( c^* \in C^* \) and \( c \in C \). The above right \( C^* \)-action on \( \text{Hom}(C, A) \) is very natural: \( \text{Hom}(C, A) \) has an algebra structure which is the right-left version of the smash product in the sense of Koppinen ([28]), i.e. the multiplication is given by

\[
(f \cdot g)(c) = \sum f\left(c(1)\right)_{<0>} g\left(c(2) \cdot f\left(c(1)\right)_{<-1>}\right)
\]  
(19)

for all \( f, g \in \text{Hom}(C, A) \), \( c \in C \). Moreover, if \( C \) is finitely generated and projective, the canonical map \( i : A \# C^* \to \text{Hom}(C, A) \) given by \( i(a \# c^*)(c) = \langle c^*, c \rangle a \) is an algebra isomorphism. Now, \( \text{Hom}(C, A) \) contains \( C^* \) as a subalgebra via \( j : C^* \to \text{Hom}(C, A) \), \( j(c^*)(c) := \langle c^*, c \rangle \), and the \( C^* \)-action given by (18) is exactly the structure induced by the usual restriction of scalars via \( j : C^* \to \text{Hom}(C, A) \).

Let \( A \) be a left \( H \)-comodule algebra and \( C = H \), viewed as a right \( H \)-module coalgebra via the multiplication on \( H \). Then \( H^{\text{op}}M(H)_A = H_M^A \), the category of (right-left) relative Hopf-modules. We can also define \( H_A^M \), the category of (left-left) relative Hopf modules: an object in this category is a \( k \)-module \( M \) which has an \( A \)-module structure and a left \( H \)-comodule structure such that the following compatibility relation holds

\[
\rho_M(am) = \sum a_{<-1>} \cdot m_{<-1>} \otimes a_{<0>} m_{<0>},
\]  
(20)

for all \( a \in A, m \in M \). If \( S \) is bijective, then \( H^{\text{op}} \) is a Hopf algebra with \( S^{-1} \) as an antipode, and there exists an equivalence of categories \( H^M_A \cong H^{\text{op}}_A^M \).

Similarly, if \( A \) is a right \( H \)-comodule algebra we can define the categories \( H_A^M \) and \( A^H_M \). For instance, an object in \( H_A^M \) is a right \( A \)-module and right \( H \)-comodule \( M \) such that the following compatibility relation holds

\[
\rho_M(ma) = \sum m_{<0>} \otimes a_{<0>} \otimes m_{<1>} a_{<1>},
\]  
(21)

for all \( a \in A, m \in M \). The category \( H_A^M \), together with all its left-right equivalent versions, was introduced in [18], and proved to be a unifying framework, at the level of Hopf algebras, for problems arising from different and apparently unrelated fields like: affine algebraic groups or more generally affine scheme, Lie algebras, compact topological groups, groups representation theory, Galois theory, Clifford theory or graded rings theory (see, for instance, [46], [47]).

### 1.2 Natural transformations versus integrals

In this section we shall present a point of view which is essential for the rest of the paper: specifically, we shall prove that the existence of an integral on a Hopf algebra is a necessary and sufficient criterion for constructing a natural transformation between two functors. This observation will allow us to define the general concept of an integral associate to a Doi-Koppinen datum \((H, A, C)\).

Let \( H \) be a Hopf algebra over a field \( k \). We recall ([48]) that a right integral on \( H \) is an element \( \varphi \in H^* \) such that \( \varphi h^* = \langle h^*, 1_H \rangle \varphi \) for all \( h^* \in H^* \). This is equivalent to the fact that \( \varphi : H \to k \) is right \( H \)-comodule map, where \( k \) has the trivial right \( H \)-comodule structure. If a right (or left) integral exists, then the antipode of \( H \) is bijective ([43]).

Doi ([19]) generalizes this concept in the obvious way as follows: let \( A \) be a right \( H \)-comodule algebra. A map \( \varphi : H \to A \) is called an integral ([19]) if \( \varphi \) is right \( H \)-colinear. Furthermore, \( \varphi \) is called a total integral if additionally \( \varphi(1_H) = 1_A \). The criterion for the existence of a total integral is given by Theorem 1.6 of [19] (we shall present only its essential part):

**Theorem 1.1** Let \( A \) be a right \( H \)-comodule algebra. The following are equivalent:
1. There exists a total integral \( \varphi : H \to A \);

2. Any Hopf module \( M \in \mathcal{M}_A^H \) is relative injective as a right \( H \)-comodule, i.e. the right \( H \)-coaction \( \rho_M : M \to M \otimes H \) splits in the category \( \mathcal{M}^H \) of right \( H \)-comodules;

3. \( \rho_A : A \to A \otimes H \) splits in the category \( \mathcal{M}^H \) of right \( H \)-comodules.

We will explain now what is behind the proof of this characterization theorem. As \((A, \rho_A) \in \mathcal{M}_A^H\), \(2) \Rightarrow 3)\) is trivial. Now, if \( \lambda_A \) is an \( H \)-colinear retraction of \( \rho_A \), then \( \varphi : H \to A \), \( \varphi(h) := \lambda_A(1_A \otimes h) \) is a total integral, i.e \(3) \Rightarrow 1)\) follows. The implication \(1) \Rightarrow 2)\) is also easy: if \( \varphi \) is a total integral then

\[ \lambda_M^\varphi : M \otimes H \to M, \quad \lambda_M^\varphi(m \otimes h) = \sum m_{<0>} \varphi(S(m_{<1>})h) \]

is an \( H \)-colinear retraction of \( \rho_M \).

There is however more to be read between the lines of this proof, and this is a main starting point for this paper. The character of \( \lambda \) is functorial: more precisely, if \( f : M \to N \) is a morphism in \( \mathcal{M}_A^H \) (i.e. \( f \) is \( A \)-linear and \( H \)-colinear), then the diagram

\[ \begin{array}{ccc}
M \otimes H & \xrightarrow{\lambda_M^\varphi} & M \\
\downarrow f \otimes \text{Id} & & \downarrow f \\
N \otimes H & \xrightarrow{\lambda_N^\varphi} & N
\end{array} \]

is commutative. Hence, \( \lambda^\varphi \) is a natural transformation

\[ \lambda^\varphi : F_A \circ (\bullet \otimes H) \circ F^H \to F_A \circ 1_{\mathcal{M}_A^H} \]

where \( F_A : \mathcal{M}_A^H \to \mathcal{M}^H \) (resp. \( F^H : \mathcal{M}_A^H \to \mathcal{M}_A \)) are the corresponding forgetful functors. Now we view the right \( H \)-coaction \( \rho \) as a natural transformation

\[ \rho : F_A \circ 1_{\mathcal{M}_A^H} \to F_A \circ (\bullet \otimes H) \circ F^H. \]

Bearing in mind the above, the theorem of Doi can be restated as follows:

**Theorem 1.2** Let \( A \) be a right \( H \)-comodule algebra. The following are equivalent:

1. there exists a total integral \( \varphi : H \to A \);

2. there exists a natural transformation \( \lambda : F_A \circ (\bullet \otimes H) \circ F^H \to F_A \circ 1_{\mathcal{M}_A^H} \) that splits \( \rho : F_A \circ 1_{\mathcal{M}_A^H} \to F_A \circ (\bullet \otimes H) \circ F^H \);

3. \( \rho_A : A \to A \otimes H \) splits in the category \( \mathcal{M}^H \) of right \( H \)-comodules.

**Remarks 1.3** 1. The above theorem is still valid leaving aside the normalizing condition \( \varphi(1_H) = 1_A \). More exactly, there exists an integral \( \varphi : H \to A \) if and only if there exists \( \lambda : F_A \circ (\bullet \otimes H) \circ F^H \to F_A \circ 1_{\mathcal{M}_A^H} \) a natural transformation. In particular, in the classic case corresponding to \( A = k \), we obtain that there exists a right integral \( \varphi : H \to k \) on \( H \) if and only if there exists a natural transformation \( \lambda : (\bullet \otimes H) \circ F^H \to 1_{\mathcal{M}_H} \). Furthermore, \( \varphi(1_H) = 1 \) if and only if \( \lambda \) splits \( \rho : 1_{\mathcal{M}_H} \to (\bullet \otimes H) \circ F^H \). This is equivalent ([10], [39]) to the fact that the forgetful functor
\( F^H : \mathcal{M}^H \to \mathcal{M}_k \) is separable, which is another way of formulating Maschke’s theorem for Hopf algebras ([48]).

2. Let \( A \) be a left \( H \)-comodule algebra. The version of Theorem 1.2 for the category \( \mathcal{H}^A \mathcal{M} \) is still true. In this case the \( H \)-colinear split of \( \rho_M : M \to H \otimes M \) associated to a left total integral \( \varphi : H \to A \) is given by the formula:

\[
\lambda_M' : H \otimes M \to M, \quad \lambda_M'(h \otimes m) = \sum \varphi(hS(m_{<1>}))m_{<0>}
\]

for all \( h \in H, m \in M \).

Now, if we deal with \( \mathcal{H}^A \mathcal{M}_A \) we have to assume that the antipode of \( H \) is bijective, in order to be able to construct a splitting for \( \rho_M \). In this case, the only possible way of constructing a left \( H \)-colinear split of \( \rho_M : M \to H \otimes M \) seems to be the one given by the formula:

\[
\lambda_M'' : H \otimes M \to M, \quad \lambda_M''(h \otimes m) = \sum m_{<0>}\varphi(S^{-1}(m_{<1>})h)
\]

for all \( h \in H, m \in M \), where \( S^{-1} \) is the inverse of \( S \). Of course, in the trivial case \( A = k \) which corresponds to classic integrals, this is not really a restriction, as the existence of a left integral on \( H \) ensures the bijectivity of the antipode ([43]). We shall see however that, even in the case \( \mathcal{H}^A \mathcal{M}_A \), the restriction ”\( S \) bijective” can be left behind (moreover, we can do the same with the condition that an antipode exists), if we replace the Doi integrals \( \varphi : H \to A \) with maps \( \gamma : H \to \text{Hom}(H, A) \). For the latter, the split of \( \rho_M \) is given by

\[
\lambda_M''(h \otimes m) = \sum m_{<0>}\gamma(h)(m_{<1>})
\]

for all \( h \in H, m \in M \).

### 1.3 Quantum Yetter-Drinfel’d modules

Let \( H \) be a Hopf algebra with a bijective antipode, \( A \) an \( H \)-bicomodule algebra, and \( C \) an \( H \)-bimodule coalgebra, this means that \( A \) is an algebra in the monoidal category \( \mathcal{H}^H \mathcal{M}_H \) of \( H \)-bicomodules and \( C \) is a coalgebra in the category \( \mathcal{H}^H \mathcal{M}_H \) of \( H \)-bimodules. The left and right \( H \)-coaction on \( A \) are denoted by \( \rho ^l : A \to H \otimes A \), \( \rho ^l(a) = \sum a_{<1>} \otimes a_{<0>} \) and \( \rho ^r : A \to A \otimes H \), \( \rho ^r(a) = \sum a_{<0>} \otimes a_{<1>} \). The three tuple \( G = (H, A, C) \) was called in [12] a Yetter-Drinfel’d datum. Let \( G = (H, A, C) \) be a Yetter-Drinfel’d datum. A representation of \( G \), also called a crossed \( G \)-module or a quantum Yetter-Drinfel’d module, is a \( k \)-module \( M \) that is at the same time a right \( A \)-module and a left \( C \)-module such that

\[
\sum m_{<1>}a_{<1>} \otimes m_{<0>} \cdot a_{<0>} = \sum a_{<1>} (m \cdot a_{<0>})_{<1>} \otimes (m \cdot a_{<0>})_{<0>}
\]  

(24)

or, equivalently,

\[
\rho_M(ma) = \sum S^{-1}(a_{<1>}) \cdot m_{<1>} \cdot a_{<1>} \otimes m_{<0>} a_{<0>}
\]

(25)

for all \( m \in M \) and \( a \in A \). The category of (right-left) crossed \( G \)-modules and \( A \)-linear, \( C \)-colinear maps will be denoted by \( \mathcal{C} \mathcal{Y} \mathcal{D}(H)_A \) and was introduced in [12]. It follows easily that, for \( C = A = H \), we obtain the classical Yetter-Drinfel’d modules \( \mathcal{H} \mathcal{Y} \mathcal{D}_H \) ([52], [44]).

In a similar way, we can introduce left-right, right-right and left-left crossed \( G \)-modules. The corresponding categories are \( \mathcal{A} \mathcal{Y} \mathcal{D}(H)^C \), \( \mathcal{Y} \mathcal{D}(H)^C \mathcal{A} \) and \( \mathcal{C} \mathcal{Y} \mathcal{D}(H) \). There exist relationships (we refer to [12] for full details) between the four types of crossed \( G \)-modules, given by the following equivalence of categories

\[
\mathcal{A} \mathcal{Y} \mathcal{D}(H)^C \cong \mathcal{C} \mathcal{Y} \mathcal{D}(H) \cong \mathcal{Y} \mathcal{D}(H^{\text{op}})^{\text{cop}} \cong \mathcal{C} \mathcal{Y} \mathcal{D}(H^{\text{op}})^{\text{cop}} \mathcal{A}^{\text{op}}.
\]
For this reason, we will focus only on the category $^{C} \mathcal{YD}(H)_A$ of right-left quantum Yetter-Drinfel’d modules. It was proved in Theorem 2.3 and Remark 2.5 of [12] that $^{C} \mathcal{YD}(H)_A$ is a special case of the category of Doi-Koppinen modules: more precisely, if $G = (H, A, C)$ is a Yetter-Drinfel’d datum, then $\mathcal{O} = (H \otimes H^{\text{op}}, A, C)$ is a Doi-Koppinen datum where $A$ is a left $H \otimes H^{\text{op}}$-comodule algebra via

$$a \rightarrow \sum (a_{<1>} \otimes S^{-1}(a_{<1>})) \otimes a_{<0>} \quad (26)$$

for all $a \in A$ and $C$ is a right $H \otimes H^{\text{op}}$-module coalgebra via

$$c \cdot (h \otimes k) = k \cdot c \cdot h \quad (27)$$

for all $c \in C$, $h, k \in H$. Then there exists an isomorphism of categories

$$^{C} \mathcal{YD}(H)_A \cong ^C \mathcal{M}(H \otimes H^{\text{op}})_A.$$

Now, we will prove the converse: the Doi-Koppinen modules category is also a special case of the quantum Yetter-Drinfel’d category, i.e. both categories are on the same level of generality. Let $\mathcal{O} = (H, A, C)$ be a Doi-Koppinen datum, that is $A$ is a left $H$-comodule algebra and $C$ is a right $H$-module coalgebra. We view $A$ as an $H$-bicomodule algebra, where the right $H$-coaction on $A$ is trivial, that is $A \rightarrow A \otimes H$, $a \rightarrow a \otimes 1$ for all $a \in A$, and $C$ as an $H$-bimodule coalgebra where the left action of $H$ on $C$ is also trivial, that is $H \otimes C \rightarrow C$, $h \otimes c \rightarrow \varepsilon(h)c$, for all $h \in H$, $c \in C$. With these structures, we can view $G = (H, A, C)$ as a Yetter-Drinfel’d datum and the compatibility condition (25) becomes exactly (2), i.e. $^C \mathcal{M}(H)_A$ is also a special case of $^{C} \mathcal{YD}(H)_A$.

Our special interest will be to correspond for the case $C = H$. For this, $^{H} \mathcal{YD}(H)_A$ will be simply denoted by $^{H} \mathcal{YD}_A$, for an arbitrary $H$-bicomodule algebra $A$. An object in this category is a $k$-module $M$ that is a right $A$-module and a left $H$-comodule such that

$$\rho_M(ma) = \sum S^{-1}(a_{<1>})m_{<1>}a_{<0>} \otimes m_{<0>}a_{<0>} \quad (28)$$

for all $m \in M, a \in A$. Now, if the right $H$-comodule structure on $A$ is trivial, that is $A \rightarrow A \otimes H$, $a \rightarrow a \otimes 1_H$, then $^{H} \mathcal{YD}_A = ^H \mathcal{M}_A$, the category of relative Hopf modules. As $^H \mathcal{M}_A$ is obtained from $^{H} \mathcal{YD}_A$ by trivializing the right coaction of $H$ on $A$, we can view the category of quantum Yetter-Drinfel’d modules $^{H} \mathcal{YD}_A$ as a quantization of the category of relative Hopf modules $^H \mathcal{M}_A$.

## 2 Total integrals of a Doi-Koppinen datum

The point of view expressed in Theorem 1.2, evidencing the fact that integrals in the sense of Doi (or classic integrals on Hopf algebras) are necessary and sufficient tools for constructing a natural transformation, leads us to the correct definition of the integrals for a Doi-Koppinen datum.

Let $M \in ^C \mathcal{M}(H)_A$ with the $C$-coaction $\rho_M : M \rightarrow C \otimes M$. We have seen in Section 1.1 that $\rho_M$ is a morphism in $^C \mathcal{M}(H)_A$, in particular in $^C \mathcal{M}$. We regarded $\rho$ as a natural transformation between the functors

$$\rho : F_A \circ 1_{^C \mathcal{M}(H)_A} \rightarrow F_A \circ (C \otimes \bullet) \circ F^C.$$

Now, in the light of the above interpretation, a total integral for a Doi-Koppinen datum should be the necessary and sufficient tool for constructing a split of $\rho$.

**Definition 2.1** Let $(H, A, C)$ be a Doi-Koppinen datum. A $k$-linear map $\gamma : C \rightarrow \text{Hom}(C, A)$ is called an integral of $(H, A, C)$ if:

$$\sum c_{(1)} \otimes \gamma(c_{(2)})(d) = \sum d_{(2)} \{\gamma(c)(d_{(1)})\}_{<1>} \otimes \{\gamma(c)(d_{(1)})\}_{<0>} \quad (29)$$
for all \( c, d \in C \). An integral \( \gamma : C \to \text{Hom}(C, A) \) is called total if
\[
\sum \gamma(c_{(1)})(c_{(2)}) = \varepsilon(c)1_A
\] (30)
for all \( c \in C \).

The concept of integral presented above is obtained by relaxing the notion of \( A \)-integral of a Doi-Koppinen datum introduced in Definition 2.6 of [10], leaving aside the \( A \)-centralizing condition. More precisely, an \( A \)-integral is a total integral \( \gamma : C \to \text{Hom}(C, A) \) satisfying the \( A \)-centralising condition
\[
\sum a_{<0>} \gamma(aca_{<-2>})(da_{<-1>}) = \gamma(c)(d)a
\]
for all \( a \in A \) and \( c, d \in C \).

The condition (30) is a normalizing condition: it can be viewed as a counterpart of the condition \( \varphi(1_H) = 1_A \), corresponding to the case \( C = H \).

The condition (29) looks very far away from the colinearity condition that appears in the case \( C = H \). However, if \( C \) is projective over \( k \), the condition (29) is in fact a colinearity condition. Let \( c^* \in C^* \). Applying \( c^* \) to the first position we obtain
\[
\sum \langle c^*, c_{(1)} \rangle \gamma(c_{(2)})(d) = \sum \langle c^*, d_{(2)} \{ \gamma(c)(d_{(1)}) \} _{<-1>} \{ \gamma(c)(d_{(1)}) \} _{<0}> \rangle
\] (31)
or equivalent, using (18)
\[
\gamma(c\cdot c^*)(d) = (\gamma(c) \cdot c^*)(d)
\]
which means that \( \gamma \) is right \( C^* \)-linear.

Furthermore, if \( C \) is projective over \( k \), then (29) is equivalent to the fact that \( \gamma \) is a right \( C^* \)-linear map. In this case, we shall go further: let \( \text{Hom}(C, A) \in \mathcal{M}_{C^*} \) with the structure from (18). We define
\[
\text{HOM}(C, A) = \text{Hom}(C, A)^\text{rat}
\]
the rational part of the right \( C^* \)-module \( \text{Hom}(C, A) \). There is another equivalent way for the definition of \( \text{HOM}(A, C) \): it is the pull-back of the following two maps:
\[
\alpha : \text{Hom}(C, A) \to \text{Hom}(C, C \otimes A), \quad \alpha(f)(c) = \sum c_{(2)} \cdot f(c_{(1)})_{<-1>} \otimes f(c_{(1)})_{<0>}
\]
and
\[
\theta : C \otimes \text{Hom}(C, A) \to \text{Hom}(C, C \otimes A), \quad \theta(c \otimes f)(d) = c \otimes f(d)
\]
for all \( c, d \in C \) and \( f \in \text{Hom}(C, A) \).

Being rational as a right \( C^* \)-module, \( \text{HOM}(C, A) \) has a natural structure of left \( C \)-comodule ([1], [49]). By definition, a \( k \)-linear map \( f : C \to A \) belongs to \( \text{HOM}(C, A) \) if and only if there exists a family of elements \( c_1, \ldots, c_n \in C \) and \( h_1, \ldots, h_n \in \text{Hom}(C, A) \) such that
\[
f \cdot c^* = \sum_{i=1}^n h_i \langle c^*, c_i \rangle
\]
which is equivalent to
\[
\sum f(d_{(1)})_{<0>} \langle c^*, d_{(2)} \cdot f(d_{(1)})_{<-1>} \rangle = \sum_{i=1}^n h_i(d) \langle c^*, c_i \rangle
\]
for all $c^* \in C^*$, $d \in C$. As $C$ is projective, the last equation is equivalent to

$$\sum_{i=1}^n c_i \otimes h_i(d) = \sum d(2) \cdot f(d(1))_{<1>} \otimes f(d(1))_{<0>}$$

for all $d \in C$.

Let now $\gamma : C \to \text{Hom}(C, A)$ be an integral of $(H, A, C)$ and $f = \gamma(c)$, $c \in C$. Then choosing $c_i = c_{(1)}$ and $h_i = \gamma(c_{(2)})$ the condition (29) assures that $\gamma(c) \in \text{HOM}(C, A)$, i.e. $\text{Im}(\gamma) \subseteq \text{HOM}(C, A)$.

We record these observations in the following

**Proposition 2.2** Let $(H, A, C)$ be a Doi-Koppinen datum such that $C$ is projective over $k$. The following statements are equivalent:

1. there exists $\gamma : C \to \text{Hom}(C, A)$ an integral of $(H, A, C)$;

2. there exists $\tilde{\gamma} : C \to \text{HOM}(C, A)$ a left $C$-comodule map.

**Remarks 2.3**

1. We shall point out now that the integrals introduced above are in line with Doi’s total integrals and with the classic integrals on Hopf algebras. For this, let $C = H$ and $\gamma : H \to \text{Hom}(H, A)$ be a total integral for $(H, A, H)$. For $c = h \in H$ and $d = 1_H$, the equation (29) takes the form

$$\sum h_{(1)} \otimes \gamma(h_{(2)})(1_H) = \sum \gamma(h)(1_H)_{<1>} \otimes \gamma(h)(1_H)_{<0>}$$

i.e the map

$$\varphi = \varphi_\gamma : H \to A, \quad \varphi(h) = \gamma(h)(1_H)$$

(32)

for all $h \in H$ is left $H$-colinear, hence a total integral.

Conversely, assume that $\varphi : H \to A$ is a total integral and that the antipode $S$ is bijective (this assumption is given by the choice of sides (right-left), according to 2) of Remark 1.3). Then

$$\gamma : H \to \text{Hom}(H, A), \quad \gamma(h)(g) = \varphi(S^{-1}(g)h)$$

(33)

for all $g, h \in H$ is a total integral for $(H, A, H)$. Indeed, for $c, d \in H$ we have

$$\sum d(2)\gamma(c)(d(1))_{<1>} \otimes \gamma(c)(d(1))_{<0>} = \sum d(2)\varphi(S^{-1}(d(1))c)_{<1>} \otimes \varphi(S^{-1}(d(1))c)_{<0>}$$

($\varphi$ is left $H$ – colinear) =

$$\sum d(2)S^{-1}(d(1))c_{(1)} \otimes \varphi(S^{-1}(d(1))c_{(2)})$$

= 

$$\sum d(3)S^{-1}(d(2))c_{(1)} \otimes \varphi(S^{-1}(d(1))c_{(2)})$$

= 

$$\sum c_{(1)} \otimes \varphi(S^{-1}(d)c_{(2)})$$

= 

$$\sum c_{(1)} \otimes \gamma(c_{(2)})(d)$$

hence $\gamma$ is a total integral in our sense.

In particular, at the level of Hopf algebras over a field $k$, the existence of an integral of $(H, k, H)$ is equivalent to the existence of a classical integral on $H$. The correspondence is given by the formulas (32) and (33). We mention that if there exists an integral $\varphi : H \to k$ on $H$, then the antipode $S$ is bijective, hence the formula (33) can be used.

2. The above general definition has an extra-bonus: it leads to the notion of an integral for a
coalgebra $C$,\footnote{For $C$ $k$-projective, the notion was introduced in Definition 4.1 of [13].} which corresponds to the Doi-Koppinen datum $(k, k, C)$. More precisely, an integral for a coalgebra $C$ is a $k$-linear map $\gamma : C \rightarrow C^*$ satisfying the condition

$$\sum c(1) \otimes \gamma(c(2))(d) = \sum d(2) \otimes \gamma(c)(d(1))$$

for all $c, d \in C$. Furthermore, $\gamma$ is called total if

$$\sum \gamma(c(1))(c(2)) = \varepsilon(c)$$

for all $c$. For a coalgebra $C$ over a field $k$, there exists a total integral $\gamma : C \rightarrow C^*$ if and only if $C$ is coseparable (Theorem 4.3 of [13]), and this is the Maschke theorem for coalgebras.

We have proved that classical integrals or Doi’s total integrals are examples of integrals in our sense. We shall indicate now two important classes of examples that are not produced in this way.

**Examples 2.4**

1. Let $C = M^n(k)$ be the $n \times n$ comatrix coalgebra, i.e. $C$ is the coalgebra with a $k$-basis $\{c_{ij} | i, j = 1, \cdots, n\}$ such that

$$\Delta(c_{ij}) = \sum_{u=1}^n c_{iu} \otimes c_{uj}, \quad \varepsilon(c_{ij}) = \delta_{ij}$$

for all $i, j = 1, \cdots, n$. Let $(H, A, C) = (H, A, M^n(k))$ be a Doi-Koppinen datum and $B = A_{co}(H) = \{ a \in A | \rho(a) = 1_H \otimes a \}$ be the subalgebra of coinvariants of $A$. Let $\mu = (\mu_{ij}) \in M_n(B)$ be an arbitrary $n \times n$-matrix over $B$ (for example, $\mu_{ij} = a_{ij}1_A$, where $a_{ij}$ are scalars of $k$). Then the map

$$\gamma = \gamma_\mu : C \rightarrow \text{Hom}(C, A), \quad \gamma(c_{ij})(c_{rs}) = \delta_{is}\mu_{rj}$$

is an integral of $(H, A, M^n(k))$. Indeed, for $c = c_{ij}$ and $d = c_{kl}$

$$\sum c(1) \otimes \gamma(c(2))(d) = \sum_{u=1}^n c_{iu} \otimes \delta_{ul}\mu_{kj} = c_{il} \otimes \mu_{kj}$$

and

$$\sum d(2)\gamma(c)(d(1))_{<1>} \otimes \gamma(c)(d(1))_{<0>} = \sum_{v=1}^n c_{vl} \gamma(c_{ij})(c_{kv})_{<1>} \otimes \gamma(c_{ij})(c_{kv})_{<0>}$$

$$(\mu_{kj} \in B) = \sum_{v=1}^n c_{vl} \otimes \delta_{iv}\mu_{kj} = c_{il} \otimes \mu_{kj}$$

i.e. $\gamma_\mu$ is an integral. On the other hand, for $c = c_{ij}$,

$$\sum c_{iu}(c_{uj}) = \sum_{u=1}^n \delta_{ij}\mu_{uu} = \delta_{ij}\text{Tr}(\mu)$$

Hence, $\gamma_\mu$ is a total integral if and only if $\text{Tr}(\mu) = 1_A$.

2. Another class of examples of total integrals arises from the graded case. Let $X$ be a set and $C = kX$ be the group-like coalgebra, i.e. $kX$ is the free $k$-module having $X$ as a basis and
Let \( \Delta(x) = x \otimes x, \varepsilon(x) = 1 \), for all \( x \in X \). Let \((H, A, C) = (H, A, kX)\) be a Doi-Koppinen datum and \( \mu = (\mu_{xy})_{x, y \in X} \) be a family of elements of \( B = A^{\text{co}(H)} \). The map

\[
\gamma = \gamma_{\mu} : kX \to \text{Hom}(kX, A), \quad \gamma(x)(y) = \delta_{xy} \mu_{xy}
\]

for all \( x, y \in X \) is an integral of \((H, A, kX)\). Indeed, for \( x, y \in X \) we have

\[
\sum y \gamma(x)(y)_{<1>} \otimes \gamma(x)(y)_{<0>} = y \otimes \delta_{xy} \mu_{xy} = x \otimes \delta_{xy} \mu_{xy} = x \otimes \gamma(x)(y)
\]

i.e. \( \gamma_{\mu} \) is an integral of \((H, A, kX)\). Furthermore, \( \gamma_{\mu} \) is a total integral iff \( \mu_{xx} = 1_A \) for all \( x \in X \).

The category \( k^X \mathcal{M}(H)_A \) associated to this Doi-Koppinen datum covers a large class of examples of \( X \)-graded representations of \( A \), starting from the category of super-graded vector spaces (corresponding to the trivial case \( H = A = k, k \) a field) to the category of graded modules by \( G \)-sets (corresponding to the case \( H = kG \), where \( G \) is a group acting on the set \( X \)).

We shall now prove that the existence of an integral \( \gamma : C \to \text{Hom}(C, A) \) permits the deformation of a \( k \)-linear map between two Doi-Koppinen modules until it becomes a \( C \)-colinear map.

**Proposition 2.5** Let \( (H, A, C) \) be a Doi-Koppinen datum, \( M \in C \mathcal{M}(H)_A, N \in C \mathcal{M} \) and \( u : N \to M \) a \( k \)-linear map. Suppose that there exists \( \gamma : C \to \text{Hom}(C, A) \) an integral. Then:

1. the map \( \tilde{u} : N \to M, \quad \tilde{u}(n) = \sum u(n_{<0>})_{<0>} \gamma(n_{<1>})(u(n_{<0>})_{<1>}) \)
   for all \( n \in N \), is left \( C \)-colinear;

2. if \( \gamma \) is a total integral and \( f : M \to N \) is a morphism in \( C \mathcal{M}(H)_A \) which is a \( k \)-split injection (resp. a \( k \)-split surjection), then \( f \) has a \( C \)-colinear retraction (resp. a section).

**Proof**

1. For \( n \in N \) we have

\[
\rho_M(\tilde{u}(n)) = \sum u(n_{<0>})_{<1>} \gamma(n_{<1>})(u(n_{<0>})_{<2>})_{<1>}
\]

\[
\otimes u(n_{<0>})_{<0>} \gamma(n_{<1>})(u(n_{<0>})_{<2>})_{<0>}
\]

\[
= \sum u(n_{<0>})_{<1>} \gamma(n_{<1>})(u(n_{<0>})_{<1>})_{<1>}
\]

\[
\otimes u(n_{<0>})_{<0>} \gamma(n_{<1>})(u(n_{<0>})_{<1>})_{<0>}
\]

\[
(29) = \sum n_{<2>} \otimes u(n_{<0>})_{<0>} \gamma(n_{<1>})(u(n_{<0>})_{<1>})
\]

\[
\sum n_{<1>} \otimes \tilde{u}(n_{<0>})
\]

\[
= (\text{Id} \otimes \tilde{u}) \rho_N(n)
\]

hence \( \tilde{u} \) is left \( C \)-colinear.

2. Let \( u : N \to M \) be a \( k \)-linear retraction (resp. section) of \( f \). Then \( \tilde{u} : N \to M \) is a left \( C \)-colinear retraction (resp. section) of \( f \). Assume first that \( u \) is a retraction of \( f \). Then, for \( m \in M \)

\[
(\tilde{u} \circ f)(m) = \sum u(f(m)_{<0>})_{<0>} \gamma(f(m)_{<1>})(u(f(m)_{<0>})_{<1>})
\]

( \( f \) is \( C \)-colinear)

\[
= \sum u(f(m_{<0>})_{<0>}) \gamma(m_{<1>})(u(f(m_{<0>})_{<1>})
\]

( \( u \circ f = \text{Id} \))

\[
= \sum m_{<0>} \gamma(m_{<2>})(m_{<1>}) = m
\]
hence \( \tilde{u} : N \rightarrow M \) is a left \( C \)-colinear retraction of \( f \). On the other hand, if \( u \) is a section of \( f \), then for \( n \in N \)

\[
(f \circ \tilde{u})(n) = \sum f\left((u(n_{<0>})_{<0>})_{<0>}\gamma(n_{<-1>})\lambda(u(n_{<0>})_{<-1>})\right)
\]

( \( f \) is \( A \)-linear) \quad \sum f\left((u(n_{<0>})_{<0>})\gamma(n_{<-1>})\lambda(u(n_{<0>})_{<-1>})\right)

( \( f \) is \( C \)-colinear) \quad \sum (f(u(n_{<0>}))_{<0>})\gamma(n_{<-1>})\left((f(u(n_{<0>}))_{<-1>})\right)\]

( \( f \circ u = \mathrm{Id} \) ) \quad \sum n_{<0>}\gamma(n_{<-2>})(n_{<-1>}) = n

i.e. \( \tilde{u} : N \rightarrow M \) is a left \( C \)-colinear section of \( f \). \( \square \)

We will prove now the version of Theorem 1.2 for Doi-Koppinen modules. We note that the counterpart of the item 3) of Theorem 1.2 has a different form: the difference is given by the fact that for \( C = H, A \in \mathcal{H}M_A \) with the natural structures, while for an arbitrary \( C, A \) does not have a structure of object in \( \mathcal{CM}(H)_A \). Thus, \( A \) is replaced now with \( C \otimes A \) and this time, \( \rho_{C \otimes A} \) splits \( C \)-bicolinearly. Parts of the proof of the following theorem are closely related to the ideas presented in Section 2 of [10].

**Theorem 2.6** Let \( (H, A, C) \) be a Doi-Koppinen datum. The following statements are equivalent:

1. there exists \( \gamma : C \rightarrow \text{Hom}(C, A) \) a total integral;
2. the natural transformation \( \rho : F_A \circ 1_{\mathcal{CM}(H)_A} \rightarrow F_A \circ (C \otimes \bullet) \circ F^C \) splits;
3. the left \( C \)-coaction on \( C \otimes A, \rho_{C \otimes A} : C \otimes A \rightarrow C \otimes C \otimes A, \rho_{C \otimes A}(c \otimes a) = \sum c(1) \otimes c(2) \otimes a \)
   splits in \( \mathcal{CM}^C \), the category of \( C \)-bicomodules.

Consequently, if one of the equivalent conditions holds, any Doi-Koppinen module is relative injective as a left \( C \)-comodule.

**Proof** 1) \( \Rightarrow \) 2) Let \( \gamma : C \rightarrow \text{Hom}(C, A) \) be a total integral. We have to construct a natural transformation \( \lambda \) that splits \( \rho \). Let \( M \in \mathcal{CM}(H)_A \) and \( u_M : C \otimes M \rightarrow M \), be the \( k \)-linear retraction of \( \rho_M : M \rightarrow C \otimes M \) given by \( u_M(c \otimes m) = \varepsilon(c)m \), for all \( c \in C \) and \( m \in M \). We define \( \lambda_M = u_M \), i.e.

\[
\lambda_M = \lambda_M(\gamma) : C \otimes M \rightarrow M, \quad \lambda_M(c \otimes m) = \sum m_{<0>}\gamma(c)(m_{<-1>})
\]

(37)

for all \( c, m \in M \). It follows from Proposition 2.5 that the map \( \lambda_M \) is a left \( C \)-colinear retraction of \( \rho_M \).

It remains to prove that \( \lambda = \lambda(\gamma) \) is a natural transformation. Let \( f : M \rightarrow N \) be a morphism in \( \mathcal{CM}(H)_A \). We have to prove that the diagram

\[
\begin{array}{ccc}
C \otimes M & \xrightarrow{\lambda_M} & M \\
\downarrow \text{Id} \otimes f & & \downarrow f \\
C \otimes N & \xrightarrow{\lambda_N} & N
\end{array}
\]
is commutative. For \( c \in C \) and \( m \in M \), using that \( f \) is right \( A \)-linear, we have
\[
(f \circ \lambda_M)(c \otimes m) = \sum f(m_{<0>} \gamma(c)(m_{<1>})) = \sum f(m_{<0>}) \gamma(c)(m_{<1>})
\]
and using that \( f \) is left \( C \)-colinear
\[
\left( \lambda_N \circ (\text{Id} \otimes f) \right)(c \otimes m) = \lambda_N(c \otimes f(m)) = \sum f(m_{<0>}) \gamma(c)(f(m_{<1>})) = \sum f(m_{<0>}) \gamma(c)(m_{<1>})
\]
i.e. \( \lambda \) is a natural transformation that splits \( \rho \).

2) \( \Rightarrow \) 3) Assume that for any \( M \in {}^C \mathcal{M}(H)_A \) the \( C \)-coaction \( \rho_M : M \to C \otimes M \) splits in the category \( {}^C \mathcal{M} \) of left \( C \)-comodules and the character of the splitting is functorial. In particular, \( \rho^C_{C \otimes A} : C \otimes A \to C \otimes C \otimes A \) splits in \( {}^C \mathcal{M} \), and let \( \lambda = \lambda^C_{C \otimes A} : C \otimes C \otimes A \to C \otimes A \) be a left \( C \)-colinear retraction of it. Using the naturality of \( \lambda \), we will prove that \( \lambda \) is also right \( C \)-colinear, where \( C \otimes A \) and \( C \otimes C \otimes A \) are right \( C \)-comodules via (7) and (10).

First, let \( V \) be a \( k \)-module and \( M \in {}^C \mathcal{M}(H)_A \). Then \( M \otimes V \in {}^C \mathcal{M}(H)_A \) via the structures arising from the ones of \( M \), i.e.
\[
(m \otimes v)a = ma \otimes v, \quad \rho_{M \otimes V} = \rho_M \otimes \text{Id}_V
\]
for all \( m \in M, a \in A \) and \( v \in V \). Using the naturality of \( \lambda \), we shall prove now that
\[
\lambda_{M \otimes V} = \lambda_M \otimes \text{Id}_V \tag{38}
\]
Let \( v \in V \) and
\[
g_v : M \to M \otimes V, \quad g_v(m) = m \otimes v
\]
Then \( g_v \) is a morphism in \( {}^C \mathcal{M}(H)_A \). From the naturality of \( \lambda \) we obtain that
\[
g_v \circ \lambda_M = \lambda_{M \otimes V} \circ (\text{Id}_C \otimes g_v)
\]
Hence
\[
\lambda_{M \otimes V}(c \otimes m \otimes v) = g_v(\lambda_M(c \otimes m)) = \lambda_M(c \otimes m) \otimes v
\]
i.e. (38) holds. In particular, let us take \( M = C \otimes A \) and \( V = C \) viewed only as a \( k \)-module. Then \( C \otimes A \otimes C \in {}^C \mathcal{M}(H)_A \) via the structures arising from the ones of \( C \otimes A \), i.e.
\[
(c \otimes a \otimes d)b = \sum cb_{<1>} \otimes ab_{<0>} \otimes d \tag{39}
\]
\[
\rho_{C \otimes A \otimes C}(c \otimes a \otimes d) = \sum c_{(1)} \otimes c_{(2)} \otimes a \otimes d \tag{40}
\]
for all \( c, d \in C, a, b \in A \). With these structures the map
\[
f = \rho^C_{C \otimes A} : C \otimes A \to C \otimes A \otimes C, \quad f(c \otimes a) = \sum c_{(1)} \otimes a_{<0>} \otimes c_{(2)}S(a_{<1>})
\]
is a morphism in \( {}^C \mathcal{M}(H)_A \). From the naturality of \( \lambda \) the following diagram
\[
\begin{array}{c}
C \otimes C \otimes A \\
\downarrow I_C \otimes f \downarrow \rho^C_{C \otimes C \otimes A} \\
C \otimes C \otimes A \otimes C
\end{array}
\]
\[
\begin{array}{ccc}
\lambda_{C \otimes A} & C \otimes C \otimes A & C \otimes A \\
C \otimes C \otimes A \otimes C & \lambda_{C \otimes A \otimes C} = \lambda_{C \otimes A} \otimes I_C & C \otimes A \otimes C
\end{array}
\]
\[
\begin{array}{cc}
\rho^C_{C \otimes A} & f = \rho^C_{C \otimes A}
\end{array}
\]
\[
\text{with these structures the map}
\]
\[
\begin{array}{c}
C \otimes C \otimes A \\
\downarrow I_C \otimes f \downarrow \rho^C_{C \otimes C \otimes A} \\
C \otimes C \otimes A \otimes C
\end{array}
\]
\[
\begin{array}{ccc}
\lambda_{C \otimes A} & C \otimes C \otimes A & C \otimes A \\
C \otimes C \otimes A \otimes C & \lambda_{C \otimes A \otimes C} = \lambda_{C \otimes A} \otimes I_C & C \otimes A \otimes C
\end{array}
\]
\[
\begin{array}{cc}
\rho^C_{C \otimes A} & f = \rho^C_{C \otimes A}
\end{array}
\]
is commutative, i.e. $\lambda = \lambda_{C \otimes A}$ is also right $C$-colinear.

3) $\Rightarrow$ 1) The left $C$-coaction $\rho^l_{C \otimes A} : C \otimes A \to C \otimes C \otimes A$ is a $C$-bicomodule map. Let $\lambda = \lambda_{C \otimes A} : C \otimes C \otimes A \to C \otimes A$ be a split of $\rho^l_{C \otimes A}$ in $C \mathcal{M}^C$. In particular,

$$\lambda(\sum c(1) \otimes c(2) \otimes a) = c \otimes a$$

for all $c \in C$, $a \in A$. We define,

$$\gamma : C \to \text{Hom}(C, A), \quad \gamma(c)(d) := (\varepsilon \otimes \text{Id})\lambda(c \otimes d \otimes 1_A) \quad (42)$$

for all $c, d \in C$. We will prove that $\gamma$ is a total integral. First,

$$\sum \gamma(c(1))(c(2)) = \sum (\varepsilon \otimes \text{Id}_A)\lambda(c(1) \otimes c(2) \otimes 1_A) = \varepsilon(c)1_A$$

i.e. (30) holds. We will prove that (29) also holds. For $c, d \in C$, the left hand side of (29) is

$$\sum c(1) \otimes \gamma(c(2))(d) = \sum c(1) \otimes (\varepsilon \otimes 1_A)\lambda(c(2) \otimes d \otimes 1_A)$$

($\lambda$ is left $C$-colinear)

$$= (\text{Id}_C \otimes \varepsilon \otimes \text{Id}_A)\rho^l_{C \otimes A}(\lambda(c \otimes d \otimes 1_A))$$

$$= \lambda(c \otimes d \otimes 1_A)$$

In order to compute the right hand side of (29) we adopt the temporary notation

$$\lambda(c \otimes d(1) \otimes 1_A) = \sum_i c_i \otimes a_i.$$ 

Now,

$$\sum d(2)\gamma(c)(d(1))_{<1>} \otimes \gamma(c)(d(1))_{<0>} = \sum d(2)\varepsilon(c_i)a_{i_{<1>}} \otimes a_{i_{<0>}}$$

$$= \sum d(2)a_{i_{<1>}} \otimes \varepsilon(c_i)a_{i_{<0>}}$$

Hence, the equation (29) is equivalent to

$$\lambda(c \otimes d \otimes 1_A) = \sum(d(2) \otimes 1_A)(\varepsilon_C \otimes \rho_A)\lambda(c \otimes d(1) \otimes 1_A) \quad (43)$$

for all $c, d \in C$. Now, it is time to use the fact that $\lambda$ is also right $C$-colinear. Denoting

$$\lambda(c \otimes d \otimes 1_A) = \sum_i D_i \otimes A_i \in C \otimes A$$

and evaluating the diagram (41) at $c \otimes d \otimes 1_A$, we obtain

$$\sum \lambda(c \otimes d(1) \otimes 1_A) \otimes d(2) = \sum D_{i(1)} \otimes A_{i_{<0>}} \otimes D_{i(2)}S(A_{i_{<1>}})$$

hence,

$$\sum d(2) \otimes (\varepsilon_C \otimes I_A)(\lambda(c \otimes d(1) \otimes 1_A)) = \sum D_iS(A_{i_{<1>}}) \otimes A_{i_{<0>}}$$

Now we apply $\rho_A$ to the second factor of both sides. Using the fact that $\rho_A \circ (\varepsilon_C \otimes I_A) = \varepsilon_C \otimes \rho_A$, we obtain

$$\sum d(2) \otimes (\varepsilon_C \otimes \rho_A)(\lambda(c \otimes d(1) \otimes 1_A)) = \sum D_iS(A_{i_{<1>}}) \otimes A_{i_{<1>}} \otimes A_{i_{<0>}}.$$ 

(43) follows after we let the second factor act on the first one. 

Leaving aside the normalizing condition (30), we obtain the following directly from the proof of the theorem:
Corollary 2.7 Let \((H, A, C)\) be a Doi-Koppinen datum. The following statements are equivalent:

1. there exists \(\gamma : C \to \text{Hom}(C, A)\) an integral of \((H, A, C)\);
2. there exists \(\lambda : F_A \circ (C \otimes \bullet) \circ F_C \to F_A \circ 1_{C, \mathcal{M}(H)_A}\) a natural transformation;
3. there exists \(\lambda' : C \otimes C \otimes A \to C \otimes A\) a \(C\)-bicomodule map.

Remark 2.8 The Theorem 2.6 has an interesting consequence in the finite dimensional case. First, let \(R \subseteq S\) be a ring extension. \(S/R\) is called a RIT-extension (right integral type) if any right \(S\)-module is injective as a right \(R\)-module. Assume that \(C\) is finite dimensional over a field \(k\) and \(\gamma : C \to \text{Hom}(C, A)\) is a total integral. Then \(C^* \subseteq A\# C^*\) is a RIT-extension. Indeed, as \(C\) is finite dimensional, the functor \(P : C, M(H)_A \to M_{A\# C^*}\) is an equivalence of categories, which allows us to apply Theorem 2.6.

The object \(C \otimes A\) plays a special role in \(C, \mathcal{M}(H)_A\): first, if \(k\) is a field, it was proved in [14] that \(C \otimes A\) is a subgenerator in \(C, \mathcal{M}(H)_A\), i.e. any object in \(C, \mathcal{M}(H)_A\) is isomorphic to a subobject of a quotient of direct sums of copies of \(C \otimes A\). Secondly, over a commutative ring, Corollary 2.9 of [11] proves that, if the forgetful functor \(F_C : C, \mathcal{M}(H)_A \to \mathcal{M}_A\) is Frobenius (i.e. by definition has the same left and right adjoint), then \(C \otimes A\) is a generator in \(C, \mathcal{M}(H)_A\). Finally, Lemma 2.8 of [17] shows that, if \(k\) is a field and \(C\) is a left and right quasi co-Frobenius coalgebra, then \(C \otimes A\) is a generator in \(C, \mathcal{M}(H)_A\). We shall prove now the main applications of the existence of a total integral.

Theorem 2.9 Let \((H, A, C)\) be a Doi-Koppinen datum and suppose that there exists \(\gamma : C \to \text{Hom}(C, A)\) a total integral. Then for any \(M \in C, \mathcal{M}(H)_A\) the map

\[
f : C \otimes A \otimes M \to M, \quad f(c \otimes a \otimes m) = \sum m_{<0>} \gamma(cS(a_{<-1>}))(m_{<-1>})a_{<0>}
\]

(44)

for all \(c \in C, a \in A\) and \(m \in M\) is a \(k\)-split epimorphism in \(C, \mathcal{M}(H)_A\).

In particular, \(C \otimes A\) is a generator in the category \(C, \mathcal{M}(H)_A\).

Proof \(C \otimes A \otimes M\) is viewed as an object in \(C, \mathcal{M}(H)_A\) with the structures arising from the ones of \(C \otimes A\), i.e.

\[
(c \otimes a \otimes m)b = \sum cb_{<-1>} \otimes ab_{<0>} \otimes m
\]

\[
\rho_{C \otimes A \otimes M}(c \otimes a \otimes m) = \sum c_{(1)} \otimes c_{(2)} \otimes a \otimes m
\]

for all \(c \in C, a, b \in A\) and \(m \in M\). First we shall prove that \(f\) is a \(k\)-split (even a \(C\)-colinear split) surjection. Let \(g : M \to C \otimes A \otimes M, g(m) = \sum m_{<-1>} \otimes 1_A \otimes m_{<0>}\), for all \(m \in M\). Then \(g\) is left \(C\)-colinear (but is not right \(A\)-linear) and for \(m \in M\) we have

\[
(f \circ g)(m) = \sum f(m_{<-1>} \otimes 1_A \otimes m_{<0>})
\]

\[
= \sum m_{<0>} \gamma(m_{<-1>})(m_{<0>}(m_{<-1>})
\]

\[
= \sum m_{<0>} \gamma(m_{<-2>})(m_{<-1>})
\]

\[
= \sum m_{<0>} \gamma(m_{<-1>(1)})(m_{<-1>(2)})
\]

\[
= \sum m_{<0>} \gamma(m_{<-1>}) = m
\]
i.e. \( g \) is a left \( C \)-colinear section of \( f \). For \( a, b \in A, c \in C \) \( m \in M \) we have

\[
 f((c \otimes a \otimes m)b) = \sum f(cb_{<1>} \otimes ab_{<0>} \otimes m)
 = \sum m_{<0>} \gamma \left( cb_{<1>} S(b_{<0>}a_{<1>}) S(a_{<1>}) \right) (m_{<1>}) a_{<0>} b_{<0>}
 = \sum m_{<0>} \gamma \left( cb_{<2>} S(b_{<1>}) S(a_{<1>}) \right) (m_{<1>}) a_{<0>} b_{<0>}
 = \sum m_{<0>} \gamma \left( cS(a_{<1>}) \right) (m_{<1>}) a_{<0>} b
 = f(c \otimes a \otimes m)b
\]

i.e. \( f \) is right \( A \)-linear. It remains to prove that \( f \) is also left \( C \)-colinear. First we compute

\[
 (IC \otimes f) \rho_{C \otimes A \otimes M}(c \otimes a \otimes m) = \sum c_{(1)} \otimes f(c_{(2)} \otimes a \otimes m)
 = \sum c_{(1)} \otimes m_{<0>} \gamma (c_{(2)} S(a_{<1>})) (m_{<1>}) a_{<0>}
\]

and

\[
 \rho_M(f(c \otimes a \otimes m)) = \sum \rho_M(m_{<0>} \gamma (cS(a_{<1>})) (m_{<1>}) a_{<0>})
 = \sum m_{<0>} \gamma \left( cS(a_{<1>}) \right) \gamma (m_{<1>}) a_{<0>}
 = \sum m_{<0>} \gamma \left( cS(a_{<1>}) \right) \gamma (m_{<1>}) a_{<0>}
 = \sum m_{<0>} \gamma \left( cS(a_{<1>}) \right) \gamma (m_{<1>}) a_{<0>}
\]

\[
 (29)
 = \sum c_{(1)} S(a_{<2>}) (1) a_{<1>} \otimes m_{<0>} \gamma (c_{(2)} S(a_{<2>}) (2)) (m_{<1>}) a_{<0>}
 = \sum c_{(1)} S(a_{<2>}) a_{<1>} \otimes m_{<0>} \gamma (c_{(2)} S(a_{<2>}) (2)) (m_{<1>}) a_{<0>}
 = \sum c_{(1)} S(a_{<2>}) a_{<1>} \otimes m_{<0>} \gamma (c_{(2)} S(a_{<2>}) (2)) (m_{<1>}) a_{<0>}
 = \sum c_{(1)} \otimes m_{<0>} \gamma (c_{(2)} S(a_{<2>}) (2)) (m_{<1>}) a_{<0>}
 = (IC \otimes f) \rho_{C \otimes A \otimes M}(c \otimes a \otimes m)
\]

i.e. \( f \) is left \( C \)-colinear. Hence, we proved that \( f \) is an epimorphism in \( C \mathcal{M}(H)_A \) and has a \( C \)-colinear section.

Now, taking a \( k \)-free presentation of \( M \) in the category of \( k \)-modules

\[
 k^{(f)} \xrightarrow{\pi} M \rightarrow 0
\]

we obtain an epimorphism in \( C \mathcal{M}(H)_A \)

\[
 (C \otimes A)^{(f)} \cong C \otimes A \otimes k^{(f)} \xrightarrow{g} M \rightarrow 0
\]

where \( g = f \circ (IC \otimes I_A \otimes \pi) \). Hence \( C \otimes A \) is a generator in \( C \mathcal{M}(H)_A \). 

\( \square \)
3 The affineness criterion for quantum Yetter-Drinfel’d modules

Let $H$ be a Hopf algebra with a bijective antipode and $A$ an $H$-bimodule algebra. Let $^{H}\mathcal{YD}_A$ be the category of (right-left) quantum Yetter-Drinfel’d modules, i.e. an object in $^{H}\mathcal{YD}_A$ is a right $A$-module, left $H$-comodule $(M,\cdot,\rho_M)$ such that
\[ \sum m_{<1>}a_{<1>} \otimes m_{<0>} \cdot a_{<0>} = \sum a_{<1>}(m \cdot a_{<0>})_{<1>} \otimes (m \cdot a_{<0>})_{<0>} \]
for all $m \in M$ and $a \in A$.

**Remark 3.1** Let $M$ be a right $A$-module and a left $H$-comodule. Then the compatibility relation (45) is equivalent to
\[ \rho_M(m \cdot a) = \sum S^{-1}(a_{<1>})m_{<1>}a_{<1>} \otimes m_{<0>} \cdot a_{<0>} \]
for all $m \in M$ and $a \in A$. Indeed, assume first that (46) holds. Then for $a \in A, m \in M$
\[ \sum a_{<1>}(m \cdot a_{<0>})_{<1>} \otimes (m \cdot a_{<0>})_{<0>} = \sum a_{<1>}S^{-1}(a_{<0>})_{<1>}m_{<1>}a_{<0>}_{<-1>} \]
\[ \otimes m_{<0>} \cdot a_{<0>} = \sum m_{<1>}a_{<-1>} \otimes m_{<0>} \cdot a_{<0>} \]

Conversely, if (45) holds then
\[ \rho_M(m \cdot a) = \sum (m \cdot a)_{<-1>} \otimes (m \cdot a)_{<0>} \]
\[ = \sum \varepsilon(a_{<1>})(m \cdot a_{<0>})_{<-1>} \otimes (m \cdot a_{<0>})_{<0>} \]
\[ = \sum S^{-1}(a_{<1>})a_{<0>}_{<1>}<(m \cdot a_{<0>})_{<0>}_{<-1>} \otimes (m \cdot a_{<0>})_{<0>}_{<0>} \]
\[ = \sum S^{-1}(a_{<1>})m_{<1>}a_{<-1>} \otimes m_{<0>} \cdot a_{<0>} \]
for all $a \in A, m \in M$.

**Examples 3.2**
1. Let $A = H$ and $\rho^l = \rho^r = \Delta$. Then $^{H}\mathcal{YD}_H$ is the category of crossed (or Yetter-Drinfel’d) modules introduced in [52] (see also [44] for all left-right conventions).
2. If $\rho_r : A \rightarrow A \otimes H$ is the trivial coaction, that is $\rho^r(a) = a \otimes 1_H$, then $^{H}\mathcal{YD}_A = ^{H}\mathcal{M}_A$, the category of classical relative Hopf modules.
3. If both coactions $\rho^l$ and $\rho^r$ are trivial, then $^{H}\mathcal{YD}_A = ^{H}\mathcal{L}_A$, the category of Long dimodules. This category was defined by F.W. Long in [31] for the case $A = H$, a commutative and cocommutative Hopf algebra and was studied in connection with the Brauer group. In the general case, the category $^{C}\mathcal{L}_A$ was introduced in [40] and was studied related to a nonlinear equation.
4. Let $H = kG$ where $G$ is a group. Then a $kG$-bimodule algebra is a bi-graded $k$-algebra $A$, having two compatible gradations of type $G$ and an object in $^{kG}\mathcal{YD}_A$ is a $G$-graded representation on $A$ such that the $A$-action agrees with the bi-gradation.

An important object of $^{H}\mathcal{YD}_A$ is the Verma structure $(A,\cdot,\tilde{\rho})$, where $\cdot$ is the multiplication on $A$ and the left $H$-coaction $\tilde{\rho}$ given by
\[ \tilde{\rho} : A \rightarrow H \otimes A, \quad \tilde{\rho}(a) = \sum S^{-1}(a_{<1>})a_{<-1>} \otimes a_{<0>} \]
for all $a \in A$. Then $(A,\cdot,\tilde{\rho}) \in ^{H}\mathcal{YD}_A$ and we will see that it will play a crucial role in this section. In the particular case $(A = H, \rho^l = \rho^r = \Delta)$, the above structure is just the right-left version of the Verma Yetter-Drinfel’d module over $H$ defined in equation (2.6) of [23].
Remark 3.3 If $H$ is commutative then $(A, \tilde{\rho})$ is a structure of left $H$-comodule algebra on $A$. Hence, we can associate the usual category $^H\mathcal{M}_A$ of classical Hopf modules to it: it is easy to show that $^H\mathcal{YD}_A = ^H\mathcal{M}_A$, i.e., in the commutative case the theory of the category of quantum Yetter-Drinfel’d modules can be reduced to the study of the Hopf modules category. The theory presented below is relevant for the noncommutative case.

We view now $^H\mathcal{YD}_A$ as the category of Doi-Koppinen modules associated to the Doi-Koppinen datum $(H \otimes H^{\operatorname{op}}, A, H)$ where

- $A$ is a left $H \otimes H^{\operatorname{op}}$-comodule algebra via
  \[
  a \to \sum (a_{< -1 >} \otimes S^{-1}(a_{< 1 >})) \otimes a_{< 0 >}
  \]
  (48)
  for all $a \in A$ and

- $H$ is a right $H \otimes H^{\operatorname{op}}$-module coalgebra via
  \[
  g \cdot (h \otimes k) = kgh
  \]
  (49)
  for all $g, h, k \in H$. Then $^H\mathcal{M}(H \otimes H^{\operatorname{op}})_A = ^H\mathcal{YD}_A$, and hence all the concepts, structures and results from previous sections can be formulated for $^H\mathcal{YD}_A$. For instance, $H \otimes A \in ^H\mathcal{YD}_A$ via the following structures

\[
\begin{align*}
(h \otimes b)a &= \sum h(a_{< -1 >})ha_{< -1 >} \otimes ba_{< 0 >} \\
\rho_{H \otimes A}(h \otimes b) &= \sum h_{(1)} \otimes h_{(2)} \otimes b
\end{align*}
\]
(50) \hspace{1cm} (51)
for all $h \in H, a, b \in A$. The Definition 2.1 has the following form

Definition 3.4 Let $H$ be a Hopf algebra with a bijective antipode and $A$ an $H$-bicomodule algebra. A $k$-linear map $\gamma : H \to \operatorname{Hom}(H, A)$ is called a quantum integral if:

\[
\sum g_{(1)} \otimes \gamma(g_{(2)})(h) = \sum S^{-1}\left(\{\gamma(g)(h_{(1)})\}_{< 1 >}\right)h_{(2)}\{\gamma(g)(h(1))\}_{< -1 >} \otimes \{\gamma(g)(h_{(1)})\}_{< 0 >}
\]
(52)
for all $g, h \in H$. A quantum integral $\gamma : H \to \operatorname{Hom}(H, A)$ is called total if

\[
\sum \gamma(h_{(1)})(h_{(2)}) = \varepsilon(h)1_A
\]
(53)
for all $h \in H$.

Remarks 3.5 1. Let $\gamma : H \to \operatorname{Hom}(H, A)$ be a quantum integral. Then,

\[
\varphi = \varphi_{\gamma} : H \to A, \quad \varphi(g) = \gamma(g)(1_H)
\]
satisfies the condition

\[
\sum g_{(1)} \otimes \varphi(g_{(2)}) = \sum S^{-1}\left(\varphi(g)_{< 1 >}\right)\varphi(g)_{< -1 >} \otimes \varphi(g)_{< 0 >}
\]
for all $g \in H$, i.e. $\varphi : H \to A$ is left $H$-colinear, where $A$ is a left $H$-comodule via $\tilde{\rho}$.

As opposed to the case $^H\mathcal{M}_A$, a left $H$-colinear map $\varphi : H \to A$ is not sufficient to construct a quantum integral. If however $\varphi : H \to A$ is a $k$-linear map satisfying the more powerful relation

\[
\sum xg_{(1)} \otimes \varphi(g_{(2)}) = \sum S^{-1}\left(\varphi(g)_{< 1 >}\right)x\varphi(g)_{< -1 >} \otimes \varphi(g)_{< 0 >}
\]
(54)
for all $g \in H$, then $\varphi : H \to A$ is a quantum integral.
for all \( x, g \in H \), then

\[
\gamma = \gamma_\varphi : H \to \text{Hom}(H, A), \quad \gamma(g)(h) = \varphi(S^{-1}(h)g)
\]

is a quantum integral. Indeed, the right hand side of (52) is

\[
\text{RHS} = \sum S^{-1}(\varphi(S^{-1}(h_{(1)})g)_{<1>})h_{(2)}\varphi(S^{-1}(h_{(1)})g)_{<-1>} \otimes \varphi(S^{-1}(h_{(1)})g)_{<0>}
\]

(54) \[
= \sum h_{(3)}S^{-1}(h_{(2)})g_{(1)} \otimes \varphi(S^{-1}(h_{(1)})g_{(2)})\\
= \sum g_{(1)} \otimes \varphi(S^{-1}(h)g_{(2)})\\
= \sum g_{(1)} \otimes \gamma(g_{(2)})(h)
\]

i.e. \( \gamma \) is a quantum integral.

2. The above remark indicates us a way to construct quantum integrals starting from integrals on \( H \). Let \( \theta : H \to k \) be a left integral on \( H \). Then \( \varphi = \varphi_\theta : H \to A, \varphi(h) = \theta(h)1_A \) satisfies (54) and hence

\[
\gamma = \gamma_\theta : H \to \text{Hom}(H, A), \quad \gamma(g)(h) = \theta(S^{-1}(h)g)1_A
\]

(55) is a quantum integral. Furthermore, if \( \theta(1_H) = 1_k \) (that is \( H \) is cosemisimple), then \( \gamma_\theta \) is a total quantum integral.

3. Assume that there exists \( \gamma : H \to \text{Hom}(H, A) \) a total quantum integral. Then any \( M \in \text{H} \mathcal{YD}_A \) is relative injective as a left \( H \)-comodule. In particular, let \( (A, \rho', \rho'') = (H, \Delta, \Delta) \), where \( H \) is a finite dimensional Hopf algebra over a field \( k \). Then the extension \( H^* \subset D(H) \) is a RIT-extension, that is any representation of the Drinfel’d double is injective as a right \( H^* \)-module.

**Proposition 3.6** Let \( H \) be a Hopf algebra with a bijective antipode and \( A \) an \( H \)-bicomodule algebra. Assume that there exists \( \gamma : H \to \text{Hom}(H, A) \) a total quantum integral. Then \( \tilde{\rho} : A \to H \otimes A \) splits in \( \text{H} \mathcal{YD}_A \).

**Proof** Using Theorem 2.6 for \( \text{H} \mathcal{M}(H \otimes H^{\text{op}})_A = \text{H} \mathcal{YD}_A \), the map

\[
\lambda : H \otimes A \to A, \quad \lambda(h \otimes a) = \sum a_{<0>}\gamma(h)(S^{-1}(a_{<1>})a_{<-1>})
\]

for all \( h \in H, a \in A \) is a left \( H \)-colinear retraction of \( \tilde{\rho} \). In particular, \( \lambda(1_H \otimes 1_A) = 1_A \) and

\[
\sum g_{(1)} \otimes \lambda(g_{(2)} \otimes a) = \sum S^{-1}(\lambda(g \otimes a)_{<1>})\lambda(g \otimes a)_{<-1>} \otimes \lambda(g \otimes a)_{<0>}
\]

(56) for all \( g \in H \) and \( a \in A \). We define now

\[
\Lambda : H \otimes A \to A, \quad \Lambda(h \otimes a) = \sum \lambda\left(S^{-2}(a_{<1>})hS(a_{<-1>}) \otimes 1_A\right)a_{<0>}
\]

(57) for all \( h \in H, a \in A \). Then, for \( a \in A \) we have

\[
(\Lambda \circ \tilde{\rho})(a) = \sum \Lambda(S^{-1}(a_{<1>})a_{<-1>} \otimes a_{<0>})
\]

\[
= \sum \lambda\left(S^{-2}(a_{<1>})S^{-1}(a_{<2>})S(a_{<-2>})a_{<-1>} \otimes 1_A\right)a_{<0>}
\]

\[
= \sum \lambda\left(S^{-1}(a_{<2>})S^{-1}(a_{<1>})S(a_{<-2>})a_{<-1>} \otimes 1_A\right)a_{<0>}
\]

\[
= \lambda(1_H \otimes 1_A)a = a
\]
i.e. $\Lambda$ is still a retraction of $\tilde{\rho}$. Now, for $h \in H$, $a, b \in A$ we have
\[
\Lambda((h \otimes a)b) = \sum \Lambda\left(S^{-1}(b_{<1>})hb_{<1>-1} \otimes ab_{<0>}\right) \\
= \sum \lambda\left(S^{-2}(a_{<1>})S^{-2}(b_{<1>})S^{-1}(b_{<2>})hb_{<2>},S(b_{<1>})S(a_{<1>}) \otimes 1_A\right)a_{<0>},b_{<0>} \\
= \sum \lambda\left(S^{-2}(a_{<1>})S^{-1}(b_{<2>}),S^{-1}(b_{<1>})hb_{<2>},S(b_{<1>})S(a_{<1>}) \otimes 1_A\right)a_{<0>},b_{<0>} \\
= \sum \lambda\left(S^{-2}(a_{<1>})hS(a_{<1>}) \otimes 1_A\right)a_{<0>},b \\
= \Lambda(h \otimes a)b
\]

hence $\Lambda$ is right $A$-linear. It remains to prove that $\Lambda$ is also left $H$-colinear:
\[
\tilde{\rho}\Lambda(h \otimes a) = \sum \tilde{\rho}\left(\lambda(S^{-2}(a_{<1>})hS(a_{<1>}) \otimes 1_A)a_{<0>}\right) \\
= \sum S^{-1}(a_{<1>})S^{-1}\left(\lambda(S^{-2}(a_{<2>})hS(a_{<2>}) \otimes 1_A)a_{<1>} \otimes \lambda(S^{-2}(a_{<2>})hS(a_{<2>}) \otimes 1_A)a_{<0>}\right) \\
= \sum S^{-1}(a_{<1>})S^{-2}(a_{<2>})h(1)aS(a_{<2>})a_{<1>} \otimes \lambda(S^{-2}(a_{<2>})hS(a_{<2>}) \otimes 1_A)a_{<0>} \\
= \sum h(1) \otimes \lambda(S^{-2}(a_{<1>})h(2)S(a_{<1>}) \otimes 1_A)a_{<0>} \\
= (1d \otimes \Lambda)\rho_{H \otimes A}(h \otimes a)
\]
i.e. we proved that $\Lambda$ is a retraction of $\tilde{\rho}$ in $H\mathcal{YD}_A$. \hfill \Box

We can define now the coinvariants of $A$ as follows:
\[
B = A^{co(H)} := \{ a \in A \mid \tilde{\rho}(a) = 1_H \otimes a \} = \{ a \in A \mid \sum S^{-1}(a_{<1>})a_{<1>-1} \otimes a_{<0>} = 1_H \otimes a \}
\]
Then $B$ is a subalgebra of $A$ and will be called the subalgebra of quantum coinvariants.

**Proposition 3.7** Let $H$ be a Hopf algebra with a bijective antipode, $A$ an $H$-bicomodule algebra and $B$ the subalgebra of quantum coinvariants. Assume there exists $\gamma : H \rightarrow Hom(H,A)$ a total quantum integral. Then:

1. $B$ is a direct summand of $A$ as a left $B$-submodule;
2. $B$ is a direct summand of $A$ as a right $B$-submodule.

**Proof** 1. We shall prove that there exists a well defined left trace given by the formula
\[
t^l : A \rightarrow B, \quad t^l(a) = \lambda(1_H \otimes a) = \sum a_{<0>},\gamma(1_H)(S^{-1}(a_{<1>}),a_{<1>}) \\
\]
for all $a \in A$. Taking $g = 1_H$ in (56) we obtain $1_H \otimes t^l(a) = \tilde{\rho}(t^l(a))$, i.e. $t^l(a) \in B$, for all $a \in A$. Now, for $b \in B$ and $a \in A$
\[
t^l(ba) = \sum ba_{<0>},\gamma(1_H)(S^{-1}(a_{<1>}),b_{<1>},a_{<1>}) \\
(\text{in } B) = \sum ba_{<0>},\gamma(1_H)(S^{-1}(a_{<1>}),a_{<1>}) \\
= bt^l(a)
\]
hence $t^l$ is a left $B$-module map and finally

$$t^l(1_A) = 1_A \gamma(1_H)(1_H) = 1_A \varepsilon(1_H) = 1_A$$

hence $t^l$ is a left $B$-module retraction of the inclusion $B \subset A$.

2. Similarly, we can prove that the map given by the formula

$$t^r : A \to B, \quad t^r(a) = \Lambda(1_H \otimes a) = \sum \gamma(S^{-2}(a_{<1>})S(a_{<-1>})(1_H)a_{<0>})$$

(59)

for all $a \in A$, is a well defined right trace of the inclusion $B \subset A$.

**Definition 3.8** Let $H$ be a Hopf algebra with a bijective antipode, $A$ an $H$-bicomodule algebra and $\gamma : H \to \text{Hom}(H, A)$ a total quantum integral. The map

$$t^l : A \to B, \quad t^l(a) = \sum a_{<0>} \gamma(1_H)(S^{-1}(a_{<1>})a_{<-1>})$$

for all $a \in A$ is called the (left) quantum trace associated to $\gamma$.

Now, we shall construct functors connecting $H \mathcal{YD}_A$ and $\mathcal{M}_B$. First, if $M \in H \mathcal{YD}_A$, then

$$M^{\co(H)} = \{ m \in M \mid \rho_M(m) = 1_H \otimes m \}$$

is the right $B$-module of the coinvariants of $M$. Furthermore, $M \to M^{\co(H)}$ gives us a covariant functor

$$(\cdot)^{\co(H)} : H \mathcal{YD}_A \to \mathcal{M}_B.$$  

Now, for $N \in \mathcal{M}_B$, $N \otimes_B A \in H \mathcal{YD}_A$ via the structures

$$(n \otimes_B a)a' = n \otimes_B aa'$$

(60)

$$\rho_{N \otimes_B A}(n \otimes_B a) = \sum S^{-1}(a_{<1>})a_{<-1>} \otimes n \otimes_B a_{<0>}$$

(61)

for all $n \in N$, $a, a' \in A$. In this way, we have constructed a covariant functor called the induction functor

$$- \otimes_B A : \mathcal{M}_B \to H \mathcal{YD}_A.$$  

We shall prove now that the above functors are an adjoint pair. In the case $A = H$ the next result is the right-left version of the Proposition 3.5 of [15].

**Proposition 3.9** Let $H$ be a Hopf algebra with a bijective antipode and $A$ an $H$-bicomodule algebra. Then the induction functor $- \otimes_B A : \mathcal{M}_B \to H \mathcal{YD}_A$ is a left adjoint of the coinvariant functor $(-)^{\co(H)} : H \mathcal{YD}_A \to \mathcal{M}_B$.

**Proof** Straightforward: the unit and the counit of the adjointness are given by

$$\eta_N : N \to (N \otimes_B A)^{\co(H)}, \quad \eta_N(n) = n \otimes_B 1_A$$

(62)

for all $N \in \mathcal{M}_B, n \in N$ and

$$\beta_M : M^{\co(H)} \otimes_B A \to M, \quad \beta_M(m \otimes_B a) = ma$$

(63)

for all $M \in H \mathcal{YD}_A, m \in M^{\co(H)}$ and $a \in A$.  

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With the structures given by (50) and (51), $H \otimes A \in {^H\mathcal{YD}}_A$ and we identify $({H \otimes A})^{co(H)} \cong A$ via $a \rightarrow 1_H \otimes a$. Then the adjunction map $\beta_{H \otimes A}$ can be viewed as a map in $H{^H\mathcal{YD}}_A$, as follows
\[
\beta = \beta_{H \otimes A} : A \otimes_B A \rightarrow H \otimes A, \quad \beta(a \otimes_B b) = \sum S^{-1}(b_{<1>})b_{<-1>} \otimes ab_{<0>}
\] (64)

for all $a, b \in A$. Here $A \otimes_B A \in H{^H\mathcal{YD}}_A$ via the structures $(a \otimes_B b)a' = a \otimes_B ba'$, $a \otimes_B b \rightarrow \sum S^{-1}(b_{<1>})b_{<-1>} \otimes a \otimes_B b_{<0>}$

for all $a, a', b \in A$.

**Definition 3.10** Let $H$ be a Hopf algebra with a bijective antipode, $A$ an $H$-bicomodule algebra and $B = A^{co(H)}$. Then $A/B$ is called a quantum Galois extension if the canonical map
\[
\beta : A \otimes_B A \rightarrow H \otimes A, \quad \beta(a \otimes_B b) = \sum S^{-1}(b_{<1>})b_{<-1>} \otimes ab_{<0>}
\]
is bijective.

If the right coaction $\rho^r : A \rightarrow A \otimes H$ is trivial, the above definition is just the left version of the usual well studied $H$-Galois extensions ([29], [41]). The quantum Galois extensions are the concepts that occur in the following imprimitivity statement for quantum Yetter-Drinfel’d modules.

**Proposition 3.11** Let $H$ be a Hopf algebra with a bijective antipode, $A$ an $H$-bicomodule algebra and $B = A^{co(H)}$. The following statements are equivalent:

1. the induction functor $- \otimes_B A : \mathcal{M}_B \rightarrow H{^H\mathcal{YD}}_A$ is an equivalence of categories;

2. the following conditions hold:
   - (a) $A$ is faithfully flat as a left $B$-module;
   - (b) $A/B$ is a quantum Galois extension.

**Proof**
1) $\Rightarrow$ 2) Trivial.
2) $\Rightarrow$ 1) is standard from the categorical point of view: a pair of adjoint functors (as $- \otimes_B A : \mathcal{M}_B \rightarrow H{^H\mathcal{YD}}_A$ and $(-)^{co(H)} : H{^H\mathcal{YD}}_A \rightarrow \mathcal{M}_B$ are) gives an equivalence of categories iff one of them is faithfully exact (or both of them are exact) and the adjunction maps in the key objects of categories ($B$ in $\mathcal{M}_B$ and $H \otimes A$ in $H{^H\mathcal{YD}}_A$) are bijective. We point out that $\eta_N$ for all $N \in \mathcal{M}_B$, and $\beta_M$ for all $M \in H{^H\mathcal{YD}}_A$, can be constructed from $\eta_B$ and $\beta_{H \otimes A}$ using the naturality condition: for details, in a more general frame, we refer to Theorem 4.9 of [9] or [14] (for Doi-Koppinen modules) or Theorem 3.10 of [6] (for entwining modules).

We are going to prove now an affineness condition for quantum Yetter-Drinfel’d modules. First we need the following

**Theorem 3.12** Let $H$ be a Hopf algebra with a bijective antipode, $A$ an $H$-bicomodule algebra and $B = A^{co(H)}$. Assume that there exists $\gamma : H \rightarrow \text{Hom}(H, A)$ a total quantum integral. Then
\[
\eta_N : N \rightarrow (N \otimes_B A)^{co(H)}, \quad \eta_N(n) = n \otimes_B 1_A
\]
is an isomorphism of right $B$-modules for all $N \in \mathcal{M}_B$. 

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Proof Using the left quantum trace \( t^i : A \to B \) we shall construct an inverse of \( \eta_N \). We define

\[
\theta_N : (N \otimes_B A)^{\text{co}(H)} \to N, \quad \theta_N(\sum_i n_i \otimes_B a_i) = \sum_i n_i t^i(a_i) \tag{65}
\]

for all \( \sum_i n_i \otimes_B a_i \in (N \otimes_B A)^{\text{co}(H)} \). As \( t^i(1_A) = 1_A, \theta_N \circ \eta_N = \text{Id}_N \). Let now, \( \sum_i n_i \otimes_B a_i \in (N \otimes_B A)^{\text{co}(H)} \). Then

\[
\sum S^{-1}(a_{i_{<1>}}) a_{i_{<0>}} \otimes n_i \otimes_B a_i = \sum 1_H \otimes n_i \otimes_B a_i.
\]

It follows that (after we apply first \( \gamma(1_H) \) and then the flip map \( \tau \))

\[
\sum n_i \otimes_B a_{i_{<0>}} \otimes \gamma(1_H) (S^{-1}(a_{i_{<1>}}) a_{i_{<0>}}) = \sum n_i \otimes_B a_i \otimes 1_A.
\]

Now, if we multiply the last factors we get

\[
\sum n_i \otimes_B t^i(a_i) = \sum n_i \otimes_B a_i.
\]

Hence we obtain

\[
(\eta_N \circ \theta_N)(\sum_i n_i \otimes_B a_i) = \sum_i n_i t^i(a_i) \otimes_B 1_A = \sum n_i \otimes_B t^i(a_i) = \sum n_i \otimes_B a_i
\]

i.e. \( \theta_N \) is an inverse of \( \eta_N \). \( \square \)

The above theorem applies in a large number of situations, as the following Corollary shows.

Corollary 3.13 Let \( A \) be an \( H \)-bicomodule algebra where \( H \) is a cosemisimple Hopf algebra over a field \( k \) and \( B = A^{\text{co}(H)} \). Then

\[
\eta_N : N \to (N \otimes_B A)^{\text{co}(H)}, \quad \eta_N(n) = n \otimes_B 1_A
\]

is an isomorphism of right \( B \)-modules for all \( N \in \mathcal{M}_B \).

Proof As \( H \) is cosemisimple, there exists a left integral \( \theta : H \to k \) on \( H \) with \( \theta(1_H) = 1_k \) ([1]) and the antipode of \( H \) is bijective. Then, using 2) of Remark 3.5, \( \gamma : H \to \text{Hom}(H, A), \gamma(g)(h) = \theta(S^{-1}(h)g)1_A \) for all \( g, h \in H \) is a total quantum integral and Theorem 3.12 applies. \( \square \)

Remark 3.14 The above Corollary was proven recently in Theorem 3.4 of [15] in the case \( (A = H, \rho^H = \rho = \Delta) \), where \( H \) is a semisimple and cosemisimple Hopf algebra. The strategy adopted in [15] for proving this result also used the semisimplicity of \( H \), which together with the cosemisimplicity assures that \( B = O(H) \) is a semisimple subalgebra of \( H \).

We shall prove now the main result of this section, that is the affineness criterion for quantum Yetter-Drinfel’d modules.

Theorem 3.15 Let \( H \) be a Hopf algebra with a bijective antipode and projective over \( k \), \( A \) an \( H \)-bicomodule algebra and \( B = A^{\text{co}(H)} \). Assume that:

1. there exists \( \gamma : H \to \text{Hom}(H, A) \) a total quantum integral;
2. the canonical map \( \beta : A \otimes_B A \to H \otimes A, \beta(a \otimes_B b) = \sum S^{-1}(b_{<1>}) b_{<0>} \otimes a b_{<0>} \) is surjective.
Then the induction functor $- \otimes_B A : \mathcal{M}_B \to ^H \mathcal{YD}_A$ is an equivalence of categories.

**Proof** In Theorem 3.12 we have shown that, under the assumption 1), the adjunction map $\eta_N : N \to (N \otimes_B A)^{\text{co}(H)}$ is an isomorphism for all $N \in \mathcal{M}_B$. It remains to prove that the other adjunction map, namely $\beta_M : M^{\text{co}(H)} \otimes_B A \to M, \beta_M(m \otimes_B a) = ma$ is an isomorphism for all $M \in ^H \mathcal{YD}_A$. For this we shall try to adapt the proof from Theorem 3.5 of [46].

Let $V$ be a $k$-module. Then $A \otimes V \in ^H \mathcal{YD}_A$ via the structures induced by $A$ i.e.

$$\rho_{A \otimes V}(a \otimes v) = \sum S^{-1}(a_{<1>})a_{<1>} \otimes a_{<0>} \otimes v$$

for all $a, b \in A$ and $v \in V$. In particular, for $V = A, A \otimes A \in ^H \mathcal{YD}_A$ via

$$\rho_{A \otimes A}(a \otimes a') = \sum S(a_{<1>})a_{<1>} \otimes a_{<0>} \otimes a'$$

for all $a, a', b \in A$. We will prove first that the adjunction map $\beta_{A \otimes V} : (A \otimes V)^{\text{co}(H)} \otimes_B A \to A \otimes V$ is an isomorphism for any $k$-module $V$.

First, $V \otimes B$ and $B \otimes V \in \mathcal{M}_B$ via the usual $B$-actions $(v \otimes b)b' = v \otimes bb'$, and $(b \otimes v)b' = bb' \otimes v$ for all $v \in V, b, b' \in B$. The flip map $\tau : V \otimes B \to B \otimes V, \tau(v \otimes b) = b \otimes v$ is an isomorphism in $\mathcal{M}_B$. On the other hand $V \otimes A \in ^H \mathcal{YD}_A$ via

$$\rho_{V \otimes A}(v \otimes a) = \sum S^{-1}(a_{<1>})a_{<1>} \otimes v \otimes a_{<0>}$$

for all $a, b \in A$ and $v \in V$. The flip map $\tau : A \otimes V \to V \otimes A, \tau(a \otimes v) = v \otimes a$, is an isomorphism in $^H \mathcal{YD}_A$. Applying Theorem 3.12 for $N = V \otimes B \cong B \otimes V$, we obtain the following isomorphisms in $\mathcal{M}_B$

$$B \otimes V \cong V \otimes B \cong (V \otimes B \otimes_B A)^{\text{co}(H)} \cong (V \otimes A)^{\text{co}(H)} \cong (A \otimes V)^{\text{co}(H)}$$

The adjunction map $\beta_{A \otimes V}$ for $A \otimes V$ is an isomorphism, as it is the composition of the canonical isomorphisms

$$(A \otimes V)^{\text{co}(H)} \otimes_B A \cong (V \otimes A)^{\text{co}(H)} \otimes_B A \cong V \otimes B \otimes_B A \cong V \otimes A \cong A \otimes V.$$ 

Let

$$\tilde{\beta} : A \otimes A \to H \otimes A, \quad \tilde{\beta}(a \otimes b) = \sum S^{-1}(b_{<1>})b_{<1>} \otimes ab_{<0>}$$

for all $a, b \in A$. As $\beta$ is surjective, $\tilde{\beta}$ is surjective, because the diagram

$$\begin{array}{ccc}
A \otimes A & \xrightarrow{\text{can}} & A \otimes_B A \\
\downarrow{\beta} & & \downarrow{\tilde{\beta}} \\
H \otimes A & \to & H \otimes A \\
\end{array}$$

is commutative, where $\text{can} : A \otimes A \to A \otimes_B A$ is the canonical surjection. Let us define now

$$\zeta : A \otimes A \to H \otimes A, \quad \zeta(a \otimes b) = (\tilde{\beta} \circ \tau)(a \otimes b) = \sum S^{-1}(a_{<1>})a_{<1>} \otimes ba_{<0>}$$

(72)
for all $a, b \in A$. The map $\zeta$ is surjective, as $\tilde{\beta}$ and $\tau$ are. We will prove that $\zeta$ is a morphism in $H \mathcal{YD}_A$, where $A \otimes A$ and $H \otimes A$ are quantum Yetter-Drinfel’d modules via (68), (69) and (50), (51). Indeed,

$$\zeta((a \otimes b)a') = \zeta(aa' \otimes b) = \sum S^{-1}(a'_{<1>})S^{-1}(a_{<1>})a_{<-1>}a'_{<-1>} \otimes ba_{<0>}a'_{<0>}$$

(50) $$= \sum(S^{-1}(a_{<1>})a_{<-1>} \otimes ba_{<0>})a' = \zeta(a \otimes b)a'$$

and

$$\rho_{H \otimes A}(\zeta(a \otimes b)) = \sum \rho_{H \otimes A}(S^{-1}(a_{<1>})a_{<-1>} \otimes ba_{<0>})$$

(51) $$= \sum S^{-1}(a_{<1>})a_{<-1>} \otimes S^{-1}(a_{<1>})a_{<-1>} \otimes ba_{<0>}$$

$$= \sum(Id \otimes \zeta)(S^{-1}(a_{<1>})a_{<-1>} \otimes a_{<0>} \otimes b)$$

$$= (Id \otimes \zeta)\rho_{A \otimes A}(a \otimes b)$$

for all $a, a', b \in A$. Hence $\zeta$ is a surjective morphism in $H \mathcal{YD}_A$.

$H$ is projective over $k$; hence $H \otimes A$ is projective as a right $A$-module, where $H \otimes A$ is a right $A$-module in the usual way, $(h \otimes a)b = h \otimes ab$, for all $h \in H$, $a, b \in A$. On the other hand, the map

$$u : H \otimes A \rightarrow H \otimes A, \quad u(h \otimes a) = \sum S^{-1}(a_{<1>})ha_{<-1>} \otimes a_{<0>}$$

is an isomorphism of right $A$-modules: here the first $H \otimes A$ has the usual right $A$-module structure and the second $H \otimes A$ has the right $A$-module structure given by (50). The $A$-linear inverse of $u$ is given by

$$u^{-1} : H \otimes A \rightarrow H \otimes A, \quad u^{-1}(h \otimes a) = \sum S^{-2}(a_{<1>})hS(a_{<-1>}) \otimes a_{<0>}$$

In fact, $u$ is the isomorphism given by (15), associated to $H \mathcal{YD}_A$. We obtain that $H \otimes A$, with the $A$-module structure given by (50), is still projective as a right $A$-module. It follows that the surjective morphism $\zeta : A \otimes A \rightarrow H \otimes A$ splits in the category of right $A$-modules. In particular, $\zeta$ is a $k$-split epimorphism in $H \mathcal{YD}_A$.

Let now $M \in H \mathcal{YD}_A$. Then $A \otimes A \otimes M \in H \mathcal{YD}_A$ via the structures arising from $A \otimes A$, that is

$$(a \otimes b \otimes m)a' = aa' \otimes b \otimes m$$

(73) $$\rho_{A \otimes A \otimes M}(a \otimes b \otimes m) = \sum S^{-1}(a_{<1>})a_{<-1>} \otimes a_{<0>} \otimes b \otimes m$$

(74) for all $a, b, a' \in A, m \in M$. On the other hand, $H \otimes A \otimes M \in H \mathcal{YD}_A$ via the structures arising from the ones of $H \otimes A$, i.e

$$(h \otimes a \otimes m)b = \sum S^{-1}(b_{<1>})hb_{<-1>} \otimes ab_{<0>} \otimes m$$

(75) $$\rho_{H \otimes A \otimes M}(h \otimes a \otimes m) = \sum h(1) \otimes h(2) \otimes a \otimes m$$

(76) for all $h \in H, a, b \in A, m \in M$. We obtain that

$$\zeta \otimes Id : A \otimes A \otimes M \rightarrow H \otimes A \otimes M$$

is a $k$-split epimorphism in $H \mathcal{YD}_A$.

Applying Theorem 2.9 for $H \mathcal{M}(H \otimes H^{op})_A = H \mathcal{YD}_A$ we obtain that the map

$$f : H \otimes A \otimes M \rightarrow M, \quad f(h \otimes a \otimes m) = \sum m_{<0>} \gamma(S^{-2}(a_{<1>})hS(a_{<-1>}))(m_{<-1>})a_{<0>}$$

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is a $k$-split epimorphism in $^H\mathcal{YD}_A$. Hence, the composition
\[ g = f \circ (\zeta \otimes \text{Id}) : A \otimes A \otimes M \to M, \quad g(a \otimes b \otimes m) = \sum m_{<0>} \gamma (S^{-2}(b_{<1>}) S(b_{<-1>})) (m_{<-1>}) b_{<0>} a \]
is a $k$-split epimorphism in $^H\mathcal{YD}_A$. We note that the structure of $A \otimes A \otimes M$ as an object in $^H\mathcal{YD}_A$ is of the form $A \otimes V$, for the $k$-module $V = A \otimes M$.

To conclude, we have constructed a $k$-split epimorphism in $^H\mathcal{YD}_A$
\[ A \otimes A \otimes M = M_1 \xrightarrow{g} M \to 0 \]
such that the adjunction map $\beta_{M_1}$ for $M_1$ is bijective. As $g$ is $k$-split and there exists a total quantum integral $\gamma : H \to \text{Hom}(H, A)$, we obtain that $g$ also splits in $^HM$. In particular, the sequence
\[ M_1^{\text{co}(H)} \xrightarrow{g^{\text{co}(H)}} M^{\text{co}(H)} \to 0 \]
is exact. Continuing the resolution with $\text{Ker}(g)$ instead of $M$, we obtain an exact sequence in $^H\mathcal{YD}_A$
\[ M_2 \to M_1 \to M \to 0 \]
which splits in $^H\mathcal{M}$ and the adjunction maps for $M_1$ and $M_2$ are bijective. Using the Five lemma we obtain that the adjunction map for $M$ is bijective. \hfill \Box

Theorem 3.15 is a quantum affineness criterion. We shall have a more complete picture of Theorem 0.1 for quantum Yetter-Drinfel’d modules after solving the following open problem:

*Let $H$ be a Hopf algebra over a field $k$ with a bijective antipode, $A$ an $H$-bicomodule algebra and $B = A^{\text{co}(H)}$. Assume that the induction functor $- \otimes_B A : \mathcal{M}_B \to ^H\mathcal{YD}_A$ is an equivalence of categories. Is there a total quantum integral $\gamma : H \to \text{Hom}(H, A)$?*

### 4 Conclusions and outlooks

We have introduced the integrals associated to a Doi-Koppinen datum $(H, A, C)$ and, as a major application, the quantum integrals associated to the category of quantum Yetter-Drinfel’d modules. The integrals of a Doi-Koppinen datum $(H, A, C)$ can be extended for an entwining structure ([7], [6]) or a weak entwining structure ([8]), for a Doi-Koppinen datum over a weak Hopf algebra ([4]) or for a Doi-Koppinen datum $(H, A, C)$ in a braided category ([2]).

The notion of quantum Galois extensions was introduced by quantizing the classic Galois extensions ([29]). The latter, in the equivalent left version, can be retrieved by trivializing the right coaction $\rho^r$. This process opens the way for quantizing the Clifford theory of representations ([47]) and the theory of classic crossed products ([3]). The quantum Galois extensions introduced in the present paper are related to quantum Yetter-Drinfel’d modules, while the quantum Galois theory appearing in [21], [24] refers to vertex operator algebras.

**Acknowledgment.** All diagrams were drawn using the "diagrams" software of Paul Taylor.

**References**


