Morphisms of Relative Hopf Modules, Smash Products and Duality

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Introduction

The aim of this paper is to extend a result of H.-J. Schneider on endomorphism rings of relative Hopf modules [17, Theorem 3.2] to the case of projective Hopf algebras with bijective antipode, and to show the connection between this extension and several duality theorems for Hopf algebras.

A consequence of this generalization is the following: if $H$ is a co-Frobenius Hopf algebra over the field $k$, $A/B$ is a Hopf-Galois extension and $M$ is a right $(A, H)$-Hopf module, then, considering the usual ring embeddings

$$\text{END}_A(M)^\# \subseteq \text{END}_A(M)^H \subseteq \#(H, \text{END}_A(M)),$$

we have the following description of the image of the smallest ring: there is an algebra isomorphism

$$\text{END}_A(M)^H^\text{rat} \simeq \text{END}_B(M) \cdot H^\text{rat}.$$

Also as a consequence of our result we get extensions of results of [2], [7], [8], as well as an easy proof for the duality theorems for co-Frobenius Hopf algebras from [20].

The point we want to make is that Schneider’s theorem may be found behind virtually all duality theorems for Hopf algebras.

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It was shown in [8] that two results of Ulbrich (which combine to give a duality theorem for finite Hopf algebras) are particular cases of a theorem of H.-J. Schneider. In the same paper, a partial infinite dimensional version of Schneider’s result was given.

After recalling in the first section some of the results we are going to use or improve, in the second section of this paper we give a full extension of Schneider’s theorem (Theorem 2.1). We derive from it some new duality theorems (Corollary 2.2) involving endomorphism rings of a relative Hopf module, Doi’s smash product and the usual smash product. Another application is Corollary 2.3, describing the situation for Hopf-Galois extensions, with emphasis on the case when $H$ is co-Frobenius. From this we get extensions of the results in [8] and of some results obtained in [7] for the case of graded rings and modules, as well as an interpretation of the big smash product as an endomorphism ring, which extends a similar one given in [9]. We remark that we are able to prove [8, Proposition 3.7] without assuming that the Weak Structure Theorem holds.

Recently, A. Van Daele and Y. H. Zhang proved in [20] the duality theorems for actions and coactions of co-Frobenius Hopf algebras. The method of proof uses a Morita context associated with the (co)action of a multiplier Hopf algebra. Our aim in the third section is to use the infinite dimensional version of Schneider’s result in order to give a very simple proof for the duality theorems for co-Frobenius Hopf algebras. This section may be read separately from the second section, as we give very simple alternate proofs for the results coming from Theorem 2.5. The important thing is, however, that the role played by Schneider’s theorem in these duality theorems is as effective as the one played in the duality theorem for finite Hopf algebras.

1 Preliminaries

A Hopf algebra $H$ over the field $k$ is called co-Frobenius if $H$ has non-zero space of (left (right) integrals. Recall that $H^{\text{rat}}$ is the unique maximal left (right) rational submodule of the left (right) $H^*$-module $H^*$, and $H$ is co-Frobenius if and only if $H^{\text{rat}} \neq 0$. If $H$ is co-Frobenius, the antipode $S$ of $H$ is bijective with inverse $S$ [15, Proposition 2]. We let $\rightarrow$ and $\leftarrow$ denote the usual left and right actions respectively of $H$ on $H^*$, i.e. $(h \rightarrow h^*)(g) = h^*(gh)$ and $(h^* \leftarrow h)(g) = h^*(hg)$. The subalgebra $H^{\text{rat}}$ of $H^*$ is an $H$-$H$-bimodule algebra under these actions. If $t$ is a left (right) integral of $H$, then $tS$ is a right (left) integral.

Unless explicitly mentioned otherwise, $H$ will be a Hopf algebra which is projective over the commutative ring $k$, $A$ will denote a right $H$-comodule
algebra with 1, with comodule structure map

$$\rho : A \longrightarrow A \otimes H, \quad \rho(a) = \sum a_0 \otimes a_1.$$  

We write $A^{coH}$ for the coinvariant subalgebra of $A$,

$$A^{coH} = \{ a \in A : \sum a_0 \otimes a_1 = a \otimes 1 \}.$$  

All maps are assumed $k$-linear, $\otimes$ means $\otimes_k$, etc. We use Sweedler’s notation throughout.

We denote by $M^H_A$ the category of right $(H, A)$-Hopf modules. An object in $M^H_A$ is a right $A$-module and right $H$-comodule $M$, such that

$$\sum (m.a)_0 \otimes (m.a)_1 = \sum m_0.a_0 \otimes m_1.a_1$$

for any $m \in M$ and $a \in A$. For $M$ a right $H$-comodule, $M^{coH}$ denotes the coinvariants of $M$, i.e. $M^{coH} = \{ m \in M \mid \sum m_0 \otimes m_1 = m \otimes 1 \}$. Maps in $M^H_A$ are $A$-module and $H$-comodule maps. $End^H_A(M)$ denotes the endomorphism ring of $M \in M^H_A$.

Recall that the right $H$-comodule algebra $A$ is called a right $H$-Galois extension of its coinvariants $A^{coH}$ if the map

$$can : A \otimes_{A^{coH}} A \longrightarrow A \otimes H, \quad can(a \otimes b) = \sum ab_0 \otimes b_1,$$

is bijective.

Further details about Hopf algebras may be found in [1], [14] and [18]. The reader is also referred to [2] and [8] for additional background information.

The first result we recall is the Schneider theorem.

**Theorem 1.1** [17, Theorem 3.2] Suppose that $k$ is a field, the right $H$-comodule algebra $A$ is an $H$-Galois extension of $B := A^{coH}$ and $H$ is finite dimensional. Then there exists an isomorphism of algebras:

$$End_A(M)^\#H^* \simeq End_B(M),$$

for any $M \in M^H_A$.  

We recall now the infinite dimensional version of Theorem 1.1 given in [8]. If $M \in M^H_A$ and $N \in \mathcal{M}_A$, then $M$ is a left $H^*$, right $B$-bimodule, therefore $\text{Hom}_B(M, N)$ is a right $H^*$-module with the action defined by:

$$(f \cdot h^*)(m) = f(h^*.m) = \sum h^*(m_1)f(m_0).$$

(1)
Proposition 1.2 [8, Proposition 3.7] Suppose that $S$ is bijective and that the extension $A/A^{coH}$ satisfies the Weak Structure Theorem (this means that for any $M \in \mathcal{M}_A^H$, the canonical morphism $\phi_M : A \otimes B^{coH} \to M$, given by $\phi_M(a \otimes x) = ax$, is an isomorphism (actually it is enough to assume that $\phi_M$ is onto for any $M$). Then the map $\phi : \text{Hom}_A(M, N) \otimes H^* \to \text{Hom}_B(M, N)$ defined by $\phi(f \otimes h^*) = f \cdot h^*$ is injective.

Let $M, N \in \mathcal{M}_A^H$, where $B$ is a right $H$-comodule algebra with $b \mapsto b \otimes 1$. Then we can consider, as in [19], $\text{HOM}_A(M, N)$ (resp. $\text{HOM}_B(M, N)$) consisting of those $f \in \text{Hom}_A(M, N)$ (resp. $f \in \text{Hom}_B(M, N)$) for which there exist $f_0 \in \text{Hom}_A(M, N)$ (resp. $f_0 \in \text{Hom}_B(M, N)$) and $f_1 \in H$ such that

$$\sum f(m_0_0 \otimes f(m_0_1)S(m_1) = \sum f_0(m) \otimes f_1.$$  

(2)

We remark that $\text{HOM}_A(M, N)$ (resp. $\text{HOM}_B(M, N)$) is the rational part of $\text{Hom}_A(M, N)$ (resp. $\text{Hom}_B(M, N)$) with respect to the left $H^*$-module structure defined by

$$(h^* \cdot f)(m) = \sum h^*(f(m_0_1)S(m_1))f(m_0_0).$$  

(3)

Since $\text{HOM}_A(M, N)$ is defined in the same way as $\text{HOM}_B(M, N)$, the first is a subcomodule of the latter.

Suppose now that $M = N$. Then recall from [19] that

$$\text{END}_A(M) = \text{HOM}_A(M, M)$$

is a right $H$-comodule algebra, with the comodule structure defined by $f \mapsto \sum f_0 \otimes f_1$. In the same way, $\text{END}_B(M)$ becomes also a right $H$-comodule algebra.

We also recall that $H^*$ is a right $H$-module algebra with the action

$$(h^* \leftarrow h)(g) = h^*(hg).$$

Consequently, we can form the right smash product $\text{END}_A(M)\# H^*$ with the multiplication:

$$(f \# h^*)(u \# g^*) = \sum fu_0 \# (h^* \leftarrow u_1)g^*.$$  

The infinite version of Theorem 1.1 given in [8] is the following:

Proposition 1.3 [8, Proposition 3.8] With the hypotheses of Proposition 1.2, the map

$$\phi : \text{END}_A(M)\# H^* \to \text{End}_B(M)$$

defined by $\phi(f \otimes h^*) = f \cdot h^*$, is an injective morphism of algebras. 

$\blacksquare$
If $A$ is a right $H$-comodule algebra, we shall denote by $\#(H, A)$ the "big" smash product defined on $\text{Hom}_k(H, A)$ by the following multiplication:

$$(f \cdot g)(h) = \sum f(g(h_2)h_1)g(h_2)_0, \quad f, g \in \#(H, A), \quad h \in H. \quad (4)$$

With this multiplication, $\#(H, A)$ is an associative ring with multiplicative identity $\varepsilon_H$, the counit of $H$ [9, Section4]. $A$ is isomorphic to a subalgebra of $\#(H, A)$ by identifying $a \in A$ with the map $h \mapsto \varepsilon(h)a$. Since the multiplication in (4) for maps from $H$ to $k$ is just convolution, we also have that $H^* = \text{Hom}(H, k)$ is a subalgebra of $\#(H, A)$, and $A\#H^*$ embeds in $\#(H, A)$ via

$$(a\#h^*)(h) = h^*(h)a.$$ 

Before presenting the connection between Theorem 1.1 and the duality results of [19], as given in [8], we recall the following

**Definition 1.4** [23, 16] We will denote by $\mathcal{M}^H_A$ the category whose objects are right $A$-modules $M$ which are also left $H$-modules and right $H$-comodules such that the following conditions hold:

i) $M$ is a left $H$, right $A$-bimodule,
ii) $M$ is a right $(A, H)$-Hopf module,
iii) $M$ is a left-right $H$-Hopf module.

The morphisms in this category are the left $H$-linear, right $A$-linear and right $H$-colinear maps.

Following [16] we will call $\mathcal{M}^H_A$ the category of two-sided $(A, H)$-Hopf modules.

**Example 1.5** Let $M \in \mathcal{M}^H_A$ and consider $H \otimes M$ with the natural structures of a left $H$-module and right $A$-module, and with right $H$-comodule structure given by

$$h \otimes m \mapsto \sum h_1 \otimes m_0 \otimes h_2 m_1.$$  

Then $H \otimes M$ is a two-sided $(A, H)$-Hopf module. In particular, $U = H \otimes A$ is a two-sided $(A, H)$-Hopf module(see [19]).

**Remark 1.6** Let $N \in \mathcal{M}_A$. Then $N \otimes H \in \mathcal{M}^H_A$ if we put

$$(n \otimes h)b = \sum nb_0 \otimes hb_1,$$

and

$$\sum(n \otimes h)_0 \otimes (n \otimes h)_1 = \sum n \otimes h_1 \otimes h_2,$$
the left $H$-module structure being the natural one.

If $M \in \mathcal{M}_A^H$, then $M \otimes H \simeq H \otimes M$ (as in Example 1.5) in $H \mathcal{M}_A^H$, the isomorphism (of right $A$-modules, left $H$-modules and right $H$-comodules) being given by

$$m \otimes h \mapsto \sum hS(m_1) \otimes m_0 \text{ and } h \otimes m \mapsto \sum m_0 \otimes hm_1.$$  

In particular, $U = H \otimes A \simeq A \otimes H$ in $H \mathcal{M}_A^H$ (see also [6]).

**Theorem 1.7** (see [19]) Let $A$ be a right $H$-comodule algebra, and $M \in H \mathcal{M}_A^H$. Then

i)([19, Theorem 2.4])

$$\text{End}^H_A(M) \# H \longrightarrow \text{END}_A(M), \ f \otimes h \mapsto f \cdot h$$

is an isomorphism of right $H$-comodule algebras.

ii)([19, Theorem 1.3]) If $H$ is finite and $U$ is like in Example 1.5, then the algebras $A \# H^*$ and $\text{End}^H_A(U)$ are isomorphic.

It was proved in [8, Corollary 2.5] that both i) (in the finite case) and ii) of the above Theorem are particular cases of Theorem 1.1.

We shall use the following result

**Lemma 1.8** ([1, Lemma 3.3.7]) If $k$ is a field, $H$ is a co-Frobenius Hopf algebra, $t$ is a left integral of $H$, and $a, b \in H$, then

$$\sum t(a_2S(b))a_1 = \sum t(aS(b_1))b_2.$$  

We end by recalling from [2] the infinite dimensional analogue of Radford’s distinguished grouplike element.

**Proposition 1.9** [2, Proposition 1.3] Let $t$ be a nonzero integral of $H$ (co-Frobenius over the field $k$). Then there is a grouplike element $g$ in $H$ such that

i) $t h^* = h^*(g)t$ for all $h^* \in H^*$,

ii) $tS = g \rightarrow t$, and $tS = t \leftarrow g$.  

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2 Schneider’s result revisited

In this section we prove the main result of this paper. It is a complete infinite analogue of Schneider’s Theorem 1.1 (recall that the Hopf algebras are projective throughout). From this result we deduce immediately Propositions 1.2 and 1.3, as well as several new results, some of them extending results from [7] on graded modules and their endomorphism rings.

Theorem 2.1 Let \( H \) be a Hopf algebra with bijective antipode \( S \), \( A \) a right \( H \)-comodule algebra, \( M \in \mathcal{M}_A^H \), \( N \in \mathcal{M}_A \). Then the following assertions hold:

i) \[ \text{Hom}_k(H, \text{Hom}_A(M, N)) \cong \text{Hom}_A^H(M \otimes H, N \otimes H) \]
as right \( H^* \)-modules.

The right \( H^* \)-module structure on \( \text{Hom}_k(H, \text{Hom}_A(M, N)) \) is given by

\[
(f \cdot h^*)(h) = f(h^* \cdot h) = \sum h^*(h_2)f(h_1),
\]

(5)

and the right \( H^* \)-module structure on \( \text{Hom}_A^H(M \otimes H, N \otimes H) \) is defined as follows:

\[
(u \cdot h^*)(m \otimes h) = \sum h^*(m_1S(h_1))u(m_0 \otimes h_2).
\]

(6)

ii) The isomorphism from i) induces an algebra embedding

\[
\#(H, \text{END}_A(M)) \longrightarrow \text{End}_A^H(M \otimes H).
\]

iii) If \( k \) is a field and \( H \) is co-Frobenius, then

\[
\text{Hom}_A(M, N) \otimes H^{*\text{rat}} \cong \text{Hom}_A^H(M \otimes H, N \otimes H) \cdot H^{*\text{rat}}.
\]

Proof: i) Define

\[
\beta : \text{Hom}_k(H, \text{Hom}_A(M, N)) \longrightarrow \text{Hom}_A^H(M \otimes H, N \otimes H)
\]

by

\[
\beta(f)(m \otimes h) = \sum f(m_1S(h_1))(m_0) \otimes h_2.
\]

We have that \( \beta(f) \) is right \( A \)-linear, because

\[
\beta(f)((m \otimes h)a) = \beta(f)(\sum ma_0 \otimes ha_1)
\]

\[
= \sum f(m_1a_1S(h_1a_2))(m_0a_0) \otimes h_2a_3
\]

\[
= \sum f(m_1a_1S(a_2)S(h_1))(m_0a_0) \otimes h_2a_3
\]

\[
= \sum f(m_1S(h_1))(m_0a_0) \otimes h_2a_1
\]

\[
= \sum f(m_1S(h_1))(m_0a_0) \otimes h_2a_1
\]

\[
= (\sum f(m_1S(h_1))(m_0) \otimes h_2)a = (\beta(f)(m \otimes h))a,
\]

(7)
and it is also right $H$-colinear, since
\[
\sum \beta(f)(m \otimes h_1) \otimes h_2 = \sum f(m_1 S(h_1))(m_0) \otimes h_2 \otimes h_3 = \sum \beta(f)(m \otimes h)_0 \otimes \beta(f)(m \otimes h)_1.
\]
Define now
\[
\alpha : \text{Hom}_A^H(M \otimes H, N \otimes H) \longrightarrow \text{Hom}_k(H, \text{Hom}_A(M, N))
\]
by
\[
\alpha(u)(h)(m) = \sum (I \otimes \varepsilon)u(m_0 \otimes S(h)m_1).
\]
For all $u \in \text{Hom}_A^H(M \otimes H, N \otimes H)$ and $h \in H$, we have
\[
\alpha(u)(h)(ma) = \sum (I \otimes \varepsilon)u(m_0a_0 \otimes S(h)m_1a_1)
= \sum (I \otimes \varepsilon)(u(m_0 \otimes S(h)m_1)a) = \alpha(u)(h)(m)a,
\]
so $\alpha(u)(h)$ is right $A$-linear.

We compute now
\[
\beta(\alpha)(m \otimes h) = \sum \alpha(u)(m_1 S(h_1))(m_0) \otimes h_2
= \sum (I \otimes \varepsilon)u(m_0 \otimes h_1 S(m_2)m_1) \otimes h_2
= \sum (I \otimes \varepsilon)u(m \otimes h_1) \otimes h_2
= \sum (I \otimes \varepsilon)u(m \otimes h)_0 \otimes u(m \otimes h)_1 \quad (u \text{ is colinear})
= u(m \otimes h),
\]

since
\[
\sum (I \otimes \varepsilon)(n_i \otimes h_i) \otimes h_i = \sum n_i \otimes h_i, \quad (7)
\]
and
\[
\alpha(\beta(f))(h)(m) = \sum (I \otimes \varepsilon)(\beta(f)(m_0 \otimes S(h)m_1))
= \sum (I \otimes \varepsilon)(f(m_1 S(m_2)h_2)(m_0) \otimes S(h_1)m_3)
= \sum (I \otimes \varepsilon)(f(h_2)(m_0) \otimes S(h_1)m_1) = f(h)(m),
\]

thus $\alpha$ and $\beta$ are inverse one to each other.

It remains to prove that $\alpha$ is right $H^*$-linear. We have
\[
\beta(\alpha(u) \cdot h^*)(m \otimes h) = \sum (\alpha(u) \cdot h^*)(m_1 S(h_1))(m_0) \otimes h_2
= \sum h^*(m_2 S(h_1))\alpha(u)(m_1 S(h_2))(m_0) \otimes h_3
= \sum h^*(m_2 S(h_1))(I \otimes \varepsilon)u(m_0 \otimes h_2 S(m_2)m_1) \otimes h_3
= \sum h^*(m_2 S(h_1))(I \otimes \varepsilon)u(m_0 \otimes h_2) \otimes h_3
= \sum h^*(m_2 S(h_1))u(m_0 \otimes h_2) \quad \text{(again from (7))},
\]

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so we proved both that $\text{Hom}_A^H(M \otimes H, N \otimes H)$ is an $H^*$-module and that $\alpha$ and $\beta$ are $H^*$-linear.

ii) We show that the restriction of $\beta$ is an algebra map.
Let $f, f' \in \#(H, \text{END}_A(M))$. We have

\[
\beta((f \cdot f')(m \otimes h)) = \sum (f \cdot f')(m_1 S(h_1))(m_0) \otimes h_2
\]

\[
= \sum f(f'(m_2 S(h_1))_1 m_1 S(h_2)) f'(m_2 S(h_1))_0(m_0) \otimes h_3
\]

\[
= \sum f(f'(m_3 S(h_1))(m_0)_1 S(m_1) m_2 S(h_2))(f'(m_3 S(h_1))(m_0)_0) \otimes h_3 \\
\text{(since } f'(m_2 S(h_1)) \in \text{END}_A(M))
\]

\[
= \beta(f)(\sum f'(m_1 S(h_1))(m_0) \otimes h_2) = \beta(f)(\beta(f')(m \otimes h)).
\]

iii) Let $\mu : \text{Hom}_A(M, N) \otimes H^* \rightarrow \text{Hom}_k(H, \text{Hom}_A(M, N))$

be the usual embedding:

\[
\mu(u \otimes h^*)(h)(m) = h^*(h)u(m).
\]

Composing this embedding with the isomorphism $\beta$ from i), we get an embedding

\[
\beta \circ \mu : \text{Hom}_A(M, N) \otimes H^* \rightarrow \text{Hom}_A^H(M \otimes H, N \otimes H),
\]

which sends $u \otimes h^*$ to $(\beta \circ \mu)(u \otimes h^*)$, where

\[
(\beta \circ \mu)(u \otimes h^*)(m \otimes h) = \sum h^*(m_1 S(h_1))u(m_0) \otimes h_2
\]

\[
= (((\beta \circ \mu)(u \otimes \epsilon)) \cdot h^*)(m \otimes h).
\]

Consequently,

\[
(\beta \circ \mu)(u \otimes h^*) = ((\beta \circ \mu)(u \otimes \epsilon)) \cdot h^*
\]

and we get that $\beta \circ \mu$ induces an embedding

\[
\text{Hom}_A(M, N) \otimes H^{* \text{rat}} \rightarrow \text{Hom}_A^H(M \otimes H, N \otimes H) \cdot H^{* \text{rat}},
\]

which we are going to show is also surjective.
Let $f \in \text{Hom}_A^H(M \otimes H, N \otimes H), h^* \in H^{* \text{rat}}$. We would like to find some $f_i \in \text{Hom}_A(M, N), h_i^* \in H^{* \text{rat}}$ such that

\[
(\beta \circ \mu)(\sum f_i \otimes h_i^*) = f \cdot h^*.
\]

This will happen if and only if

\[
\alpha(f \cdot h^*) = \mu(\sum f_i \otimes h_i^*),
\]
i.e.
\[ \alpha(f \cdot h^*)(h)(m) = \mu(\sum f_i \otimes h_i^*)(h)(m) = \sum h_i^*(h)f_i(m), \]
for all \( h \in H, \ m \in M \).

Since \( h^* \in H^{\text{rat}} \), we have that the space \( \{ \sum h^*(h_2)\mathcal{S}(h_1) \mid h \in H \} \) is finite dimensional. If \( \{e_1, e_2, \ldots, e_n\} \) is a basis in this space, then we can write
\[ \sum h^*(h_2)\mathcal{S}(h_1) = \sum h_i^*(h)e_i, \]
for some \( h_i^* \in H^{\text{rat}} \). Then we have
\[ \alpha(f \cdot h^*)(h)(m) = \sum (I \otimes \varepsilon)(f \cdot h^*)(m_0 \otimes \mathcal{S}(h)m_1) \]
\[ = \sum h^*(m_1\mathcal{S}(m_2)h_2)(I \otimes \varepsilon)f(m_0 \otimes \mathcal{S}(h_1)m_3) \]
\[ = \sum h^*(h)(I \otimes \varepsilon)f(m_0 \otimes e_i m_1) = \sum h_i^*(h)f_i(m), \]
where \( f_i(m) = \sum (I \otimes \varepsilon)f(m_0 \otimes e_i m_1) \), which is right \( A \)-linear since
\[ (I \otimes \varepsilon)(\sum n_i \otimes a_i)a = (I \otimes \varepsilon)(\sum n_i a_0 \otimes a_i a_1) \]
\[ = \sum n_i \varepsilon(a_i)a = (I \otimes \varepsilon)(\sum n_i \otimes a_i)a, \]
and the proof is complete.

We give now some consequences of Theorem 2.1. The first one is a new duality theorem for infinite Hopf algebras.

**Corollary 2.2** Let \( H \) be a Hopf algebra with bijective antipode, \( A \) a right \( H \)-comodule algebra, and \( M \in \mathcal{M}_A^H \) which is finitely generated as an \( A \)-module. Then the following assertions hold:

i) We have algebra isomorphisms
\[ \#(H, \text{End}_A(M)) \simeq \text{End}_A^H(M \otimes H), \]
and
\[ \#(H, \text{End}_A(M))\#H \simeq \text{END}_A(M \otimes H). \]

ii) Assume moreover that \( M \in \mathcal{M}_A^H \). Then we have an algebra isomorphism
\[ \#(H, \text{End}_A^H(M)\#H) \simeq \text{End}_A(M \otimes H). \]

**Proof:** i) Since \( M \) is finitely generated, \( \text{END}_A(M) = \text{End}_A(M) \), and the first assertion follows directly from Theorem 2.1. The second isomorphism follows from the first one, using Example 1.5, Remark 1.6 and Theorem 1.7 i).

ii) follows from the first isomorphism of i) and Theorem 1.7 i).
We show now that Propositions 1.2 and 1.3 may be also obtained (and extended) using Theorem 2.1. Moreover, the following result will provide a characterization of the image of $\text{END}_A(M)\#H^{*\text{rat}}$ through $\phi$ from Proposition 1.3.

**Corollary 2.3** Let $H$ be a Hopf algebra with bijective antipode, $A$ a right $H$-comodule algebra, $B = A^\text{coH}$, such that $A/B$ is an $H$-Galois extension. Let $M \in \mathcal{M}_A^H$ and $N \in \mathcal{M}_A$. Then the following assertions hold:

1. We have $\text{Hom}_k(H, \text{Hom}_A(M, N)) \simeq \text{Hom}_B(M, N)$,

   $f \mapsto m \mapsto \sum f(m_1)m_0$

   as right $H^*$-modules, where $\text{Hom}_k(H, \text{Hom}_A(M, N))$ is a right $H^*$-module as in (5) and the right $H^*$-module structure on $\text{Hom}_B(M, N)$ is given by (1).

2. The isomorphism from 1) induces an algebra embedding

   $\#(H, \text{END}_A(M)) \rightarrow \text{End}_B(M)$.

3. (see [9, 4.5])

   $\#(H, A) \simeq \text{End}_B(A)$.

4. If $k$ is a field and $H$ is co-Frobenius, then

   $\text{Hom}_A(M, N) \otimes H^{*\text{rat}} \simeq \text{Hom}_B(M, N) \cdot H^{*\text{rat}}$,

   $u \otimes h^* \mapsto m \mapsto \sum h^*(m_1)u(m_0)$.

**Proof:**

1) We have that the map

$$\text{Hom}_A^H(M \otimes H, N \otimes H) \xrightarrow{\nu} \text{Hom}_B(M, N),$$

sending a map to its restriction to $(M \otimes H)^\text{coH} = M \otimes 1 \simeq M$ is an isomorphism of right $H^*$-modules, as it may be easily seen from the definition of the $H^*$-module structures.

We note for further use that the inverse of $\nu$, since $A/B$ is a Galois extension, sends $f \in \text{Hom}_B(M, N)$ to $\overline{f} \in \text{Hom}_A^H(M \otimes H, N \otimes H)$, where

$$\overline{f}(m \otimes h) = \sum f(mr_i(h))l_i(h)_0 \otimes l_i(h)_1,$$

(here $\sum r_i(h) \otimes l_i(h) = \sum r_i(h)l_i(h)_0 \otimes l_i(h)_1 = 1 \otimes h$).

2) If $M = N$, then (8) yields an algebra isomorphism

$$\text{End}_A^H(M \otimes H) \simeq \text{End}_B(M).$$

3) follows from 2) for $M = A$.

4) is clear, but, since we will need it soon, we write down explicitly the
preimage of some $f \cdot h^*$, $f \in \text{Hom}_H(M,N)$, $h^* \in H^{\text{rat}}$. By the proof of Theorem 2.1 iii), we have $f \cdot h^* = \sum f_i \cdot h_i^*$, where $f_i$ has the form

$$f_i(m) = \sum (I \otimes \varepsilon)(m_o \otimes e m_1), \quad (10)$$

for some $e \in H$. Since the Weak Structure Theorem holds by [2, Theorem 3.1], we can write an $m \in M$ as $m = \sum m_j a_j$, $m_j \in M^{\text{coh}}$, and using (9), we get

$$f_i(m) = \sum f_i(m_j a_j)$$
$$= \sum f((m_j a_j)_1 r_i(e(m_j a_j)_1)) l_i(e(m_j a_j)_1) \text{(by } (9),(10)\text{)}$$
$$= \sum f(m_j a_j r_i(a_j_1) r_k(e)) l_k(e) l_i(a_j_1)$$
$$= \sum f(m_j r_i(a_j) r_k(e)) l_k(e) a_j,$$

since $\sum a_0 r_i(a_1) \otimes l_i(a_1) = 1 \otimes a$ (this is checked by applying can).

**Remarks 2.4**

1) Taking $M = N = A$ in Theorem 2.1 i), and taking into account the Remark 1.6, we obtain the isomorphism given by Ulbrich in [19, Lemma 1.2], composed with $\text{Hom}_k(S,A)$. This is a result of the type of [12, Proposition 4.1]: to see this, consider first $H \otimes A$ as an object in $\mathbb{A}^{\mathbb{M}^H}$ via

$$a(h \otimes b) = \sum a_1 h \otimes a_0 b \text{ and } h \otimes b \mapsto \sum h_1 \otimes b \otimes h_2.$$

Consequently, by [12, (3.2), p.66], $H \otimes A$ is a left $\#(H,A)$-module by

$$f : (h \otimes a) = \sum f(h_2)_1 h_1 \otimes f(h_2)_0 b. \quad (11)$$

Now $A$ is a right $H^{\text{cop}}$-comodule algebra via

$$a \mapsto \sum a_0 \otimes S(a_1),$$

we denote it by $A'$. In this way,

$$\#(H,A) \longrightarrow \#(H^{\text{cop}},A'), \quad f \mapsto f S$$

is a ring isomorphism, and its composition with Ulbrich’s isomorphism, regarded as

$$\#(H^{\text{cop}},A') \longrightarrow \text{Hom}_A^H(H \otimes A),$$

is given exactly by the left module structure as in (11).

2) Theorem 2.1 i) extends Theorem 4.3 of [7].

3) From Corollary 2.3 i) we get Proposition 1.2, under the weaker hypothesis that the extension is Galois. Note that we also obtain, when $H$ is co-Frobenius, the description of the image of $\text{Hom}_A(M,N)\#H^{\text{rat}}$ in Corollary 2.3 iii).
4) If \( M = N \), then we get Proposition 1.3 from Corollary 2.3 ii). This extends Theorem 4.8 of [7].
5) Suppose that \( k \) is a field, the antipode of \( H \) is bijective and let \( A'\#_\sigma H \) be a crossed product with invertible cocycle \( \sigma \) (see [5] or [10]). Then we have an algebra isomorphism

\[
\#(H, A'\#_\sigma H) \simeq RFM_H(A'),
\]

where \( RFM_H(A') \) are the row finite matrices, with rows and columns indexed by a basis of \( H \), and with entries in \( A' \). Indeed, we know from [4] that \( A'\#_\sigma H/A' \) is a Galois extension. Then we get from Corollary 2.3 iii) that

\[
\#(H, A'\#_\sigma H) \simeq \text{End}_{A'}(A'\#_\sigma H),
\]

and since the antipode is bijective, \( A'\#_\sigma H \) is free as a right \( A' \)-module (see [4]), and so the result follows.

This is in fact a version of [11, Proposition 4.1], which, as remarked in [11], can be also deduced from [9, 4.5] and [3, 1.18]. What it shows is that Schneider’s theorem, in the form of Theorem 2.1, is also present in the proof of the duality theorems for infinite Hopf algebras. In fact, this may also be seen in [3], where the proof of the duality theorem used embeddings in an endomorphism ring.

The following result improves Proposition 1.3 in the co-Frobenius case by describing the image of \( \text{END}_A(M)\#H^{*\text{rat}} \) in \( \text{End}_B(M) \).

**Theorem 2.5** If \( k \) is a field, \( A/B \) is Galois, \( H \) is co-Frobenius, \( M \in \mathcal{M}_A^H \) and \( N \in \mathcal{M}_A \), then

\[
\text{HOM}_A(M, N) \otimes H^{*\text{rat}} \simeq \text{HOM}_B(M, N) \cdot H^{*\text{rat}}.
\]

We also have a ring isomorphism

\[
\text{END}_A(M)\#H^{*\text{rat}} \simeq \text{END}_B(M) \cdot H^{*\text{rat}}.
\]

**Proof:** We know from Corollary 2.3 iv) that the map

\[
\text{Hom}_A(M, N) \otimes H^{*\text{rat}} \rightarrow \text{Hom}_B(M, N) \cdot H^{*\text{rat}}, \quad f \otimes h^* \mapsto f \cdot h^*
\]

is a bijection, and the restriction to \( \text{HOM}_A(M, N) \otimes H^{*\text{rat}} \) is clearly sent to \( \text{HOM}_B(M, N) \cdot H^{*\text{rat}} \). We have to show only that for any \( f \in \text{HOM}_B(M, N) \) and \( h^* \in H^{*\text{rat}} \), \( f \cdot h^* \) is in the image. We know from the proof of Corollary 2.3 iv) that \( f \cdot h^* = \sum f_i \cdot h_i^* \), where the \( f_i \in \text{Hom}_A(M, N) \) have the form

\[
\tilde{f}(m) = \sum f(m_j r_i(e)) l_i(e) a_j.
\]
We know that there are $f_0 \in \text{Hom}_B(M, N)$ and $f_1 \in H$ satisfying (2), and we want to prove that any map of the form $\tilde{f}$ has the same property, i.e. belongs to $\text{HOM}_A(M, N)$. We have

$$\sum \tilde{f}(m_0) \otimes \tilde{f}(m_0) S(m_1) =$$

$$= \sum f(m_j r_i e) a_0 l_i e a_j \otimes f(m_j r_i e) l_i e a_{j_2}$$

$$= \sum f(m_j r_i e) a_0 l_i e a_j \otimes f(m_j r_i e) l_i e 1$$

$$= \sum f(m_j r_i e) a_0 l_i e a_j \otimes f(m_j r_i e) l_i e 1$$

$$= \sum f_0 (m_j r_i e) a_0 l_i e a_j \otimes f_1 r_i e l_i e 1 = \sum \tilde{f}_0 (m) \otimes \tilde{f}_1,$$

and since $\tilde{f}_0$ is $A$-linear the proof of the first assertion is complete. The other assertion is now clear.

**Remark 2.6** When $H$ is cocommutative it is easy to see that $\text{Hom}_A(M, N)$ and $\text{Hom}_B(M, N)$ are $H^*$-bimodules with the left module structure given by (3) and the right module structure given by (1). In this case we obtain that, if $H$ is co-Frobenius, then the image of $\text{HOM}_A(M, N) \otimes H^*$ contains $\text{HOM}_B(M, N)$, similar to the graded case (see [7, Theorem 4.3]).

### 3 Application to the Duality Theorems for Co-Frobenius Hopf Algebras

Throughout this section, $H$ will be a co-Frobenius Hopf algebra over the field $k$. In this section we show that it is possible to derive the duality theorems for co-Frobenius Hopf algebras from Theorem 2.5.

We remark first that if $A$ is a right $H$-comodule algebra, then

$$A \xrightarrow{\sim} \text{END}_A(A), \quad a \mapsto f_a, \quad f_a(b) = ab$$

is an isomorphism of $H$-comodule algebras.

Indeed, if $h^* \in H^*$ and $a, b \in A$, then

$$(h^* \cdot f_a)(b) = \sum f_a(b_0) h^* (f_a(b_0) S(b_1))$$

$$= \sum a_0 b_0 h^* (a_1 b_1 S(b_2))$$

$$= \sum a_0 b_0 h^* (a_1) = \sum h^* (a_1) f_{a_0}(b)$$

which proves the desired isomorphism. Identifying $A$ with $\text{END}_A(A)$ via this isomorphism, the map $\phi$ from Proposition 1.3 becomes

$$\pi : A \# H^* \rightarrow End(A_{A^\text{com}})^{\text{rat}},$$
\[
\pi(a \# h^*)(b) = \sum ab_0h^*(b_1),
\]
which is exactly the map \(\pi_1\) from Corollary 3.4 of [2]. With this notation we have the following immediate consequence of Corollary 2.3 iv).

**Proposition 3.1** If \(A/B\) is Galois, then the map \(\pi\) is a ring isomorphism. \(\blacksquare\)

**Remark 3.2** It is also possible to derive Proposition 3.1 directly from Proposition 1.3. We know from the above remarks and from Proposition 1.3 that \(\pi\) is an injective ring morphism. (Note also that this was proved directly in [2, Corollary 3.4].) Thus it remains to show that \(\pi\) is surjective. Let \(f \in \text{End}(A_{A_{coH}})_{\text{rat}}\), and let \(h^* \in H^*_{\text{rat}}\) such that \(f = f \cdot h^*.\) Let \(t'\) be a right integral and \(\sum a_i \otimes b_i \in A \otimes_{A_{coH}} A\) such that

\[
1 \otimes h^* S = \sum a_i b_{i_0} \otimes S(b_{i_1}) \rightarrow t'.
\]

Then, for any \(b \in A\) we have

\[
\begin{align*}
 f(b) &= (f \cdot h^*)(b) \\
 &= \sum h^*(b_1)f(b_0) \\
 &= \sum h^* S(S(b_1)f(b_0)) \\
 &= \sum f(a_i b_{i_0} b_0)t'(S(b_1)S(b_{i_1})) \\
 &= \sum f(a_i b_{i_0} b_0)t'S(b_{i_1}b_1) \\
 &= \sum f(a_i b_{i_0} b_0)((S(b_{i_1}) \rightarrow t')S(b_1)) \\
 &= \pi(\sum f(a_i b_{i_0} \#(S(b_{i_1}) \rightarrow t')S(b_1))
\end{align*}
\]

and the proof is complete. \(\blacksquare\)

From Proposition 3.1 we obtain immediately the duality theorem for actions of co-Frobenius Hopf algebras.

**Corollary 3.3** [20, Theorem 5.3] If \(A' \#_{\sigma} H\) is a crossed product with invertible \(\sigma\), then we have an isomorphism of algebras

\[
(A' \#_{\sigma} H) \# H^*_{\text{rat}} \simeq M^f_H(A'),
\]

where \(M^f_H(A')\) denotes the ring of matrices with rows and columns indexed by a basis of \(H\), and with only finitely many non-zero entries in \(A'\).
Proof: The right $H^*$-module structure on $End(A'\#_s H_A)$ is given by the left $H^*$-module structure on $A'\#_s H$: $h^* \cdot (a\# h) = \sum a\# h_1 h^*(h_2)$. It is well known from [4, Theorem 1.18] that $A'\#_s H \simeq H \otimes A'$ as right $A'$-modules and left $H^*$-modules via

$$a\# h \mapsto \sum h_4 \otimes (\mathcal{S}(h_1) \cdot a) \sigma(\mathcal{S}(h_2), h_1),$$

and so $A'\#_s H_A'$ is free. Then $End(A'\#_s H_A')$ is the ring of row-finite matrices with entries in $A'$ and rows and columns indexed by a basis of $H$. In order to prove the assertion in the statement it is enough to prove that any $f$ with $f(h_i \otimes 1) \neq 0$ and $f(h_j \otimes 1) = 0$ for $j \neq i$ is in the rational part. Say $h_i \in E(S_\lambda)$. Then it is easy to see that $f = f \cdot p_\lambda$.

The converse inclusion is clear, since for any $f$ and any $p_\lambda$, $f \cdot p_\lambda$ is 0 outside $E(S_\lambda)$. 

We have seen how the first duality theorem for co-Frobenius Hopf algebras may be derived from Proposition 3.1. There is yet another, even easier way to prove this duality theorem. We sketch it since it provides a hint on how to attack the second duality theorem. It is based on the following simple

Lemma 3.4 Let $R$ be a ring without unit and $M_R$ a right $R$-module with the property that there exists a common unit (in $R$) for any finite number of elements in $R$ and $M$. Then

$$M_R \simeq Hom_R(R_R, M_R) \cdot R,$$

where, for $f \in Hom_R(R_R, M_R)$ and $r, s \in R$, $(f \cdot r)(s) = f(rs)$.

Proof: The isomorphism sends $m \in M$ to $\rho_m(r) = mr$, for all $r \in R$. The inverse sends $\sum f_i \cdot r_i$ to $\sum f_i(r_i)$. 

Corollary 3.5 If $R$ and $M$ are as in Lemma 3.4 and $M_R \simeq R_R$, then

$$R \simeq End(M_R) \cdot R$$

(as rings.)
Lemma 3.6 Let $H \otimes A$ be the two-sided Hopf module from Example 1.5. Then
\[ H \otimes A \simeq A \# H^{\text{rat}} \]
as right $A \# H^{\text{rat}}$-modules, via
\[ h \otimes a \mapsto \sum a_0 \# t \leftarrow h a_1, \]
where $t$ is a right integral of $H$.

Proof: Recall that $H \otimes A$ becomes a right $A \# H^{\text{rat}}$-module via
\[ (h \otimes a)(b \# h^*) = \sum h_1 \otimes a_0 b_0 h^*(S(h_2 a_1 b_1)). \]

If we denote by $\theta$ the map in the statement, which is clearly a bijection, then we have to show that
\[ \theta((h \otimes a)(b \# h^*)) = (\theta(h \otimes a))(b \# h^*), \]
i.e. that
\[ \sum a_0 b_0 \# t \leftarrow h_1 a_1 b_1 h^*(S(h_2 a_2 b_2)) = \sum a_0 b_0 \# (t \leftarrow h a_1 b_1) h^*. \]

If we apply the second parts to an element $x$, and denote $y = h a_1 b_1$ and $z = S(y)$, then this gets down to
\[ \sum t(S(z_2) x_1) x_2 = \sum t(S(z_1) x_1) x_2, \]
which is exactly Lemma 1.8 applied to $H^{\text{op cop}}$.

As a consequence of the above we obtain the infinite dimensional version of Theorem 1.7 ii):

Corollary 3.7 If $H$ is co-Frobenius and $A$ is a right $H$-comodule algebra, then
\[ A \# H^{\text{rat}} \simeq \text{End}_{A \# H^{\text{rat}}}(H \otimes A) \# H^{\text{rat}} = \text{End}^H_A(H \otimes A) \cdot H^{\text{rat}}. \]

Let us describe now explicitly the left $H^*$-module structure on $H \otimes A$ obtained via the isomorphism $\theta$ of Lemma 3.6. This is given by
\[ h^* \cdot (h \otimes a) = \sum h^*(S(h_1) g^{-1} h_2) \otimes a \quad (12) \]
where $h^* \in H^*$, $h \in H$, $a \in A$ and $g$ is the distinguished grouplike element from Proposition 1.9.

Indeed, we have

$$(1 \# h^*)\theta(h \otimes a) = (1 \# h^*) \left( \sum a_0 \# t \leftarrow ha_1 \right)$$

$$= \sum a_0 \# (h^* \leftarrow a_1)(t \leftarrow ha_2)$$

$$= \sum a_0 \# h^*(S(h_1)g^{-1})t \leftarrow h_2a_1$$

$$= \theta(\sum h^*(S(h_1)g^{-1})h_2 \otimes a),$$

since we have, for all $h, x \in H$

$$\sum h^*(S(h_1)g^{-1})t \leftarrow h_2x = \sum (h^* \leftarrow x_1)(t \leftarrow hx_2).$$

The last equality may be checked as follows. Apply both sides to an element $w \in H$ and denote $y = xw$ to get

$$\sum h^*(S(h_1)g^{-1})t(h_2y) = \sum h^*(y_1)t(hy_2).$$

Now denote $u = S(gh)$ and $t' = t \leftarrow g^{-1}$ to obtain

$$\sum t'(S(u_1)y)u_2 = \sum t'(S(u)y_2)y_1,$$

which is exactly Lemma 1.8 applied to $H^{op}$, for which $t'$ is still a left integral. This proves (12).

**Remark 3.8** Corollary 3.7 should be compared to Theorem 2.1 iii), noting that the right $H^*$-modules $\text{End}_H^H(H \otimes A)$ (the structure being given by (12)) and $\text{End}_A^H(A \otimes H)$ (as in (6)) are isomorphic.

Indeed, if we denote by $\tau : A \otimes H \longrightarrow H \otimes A$ the isomorphism from Remark 1.6, and if we define

$$\xi : \text{End}_H^H(A \otimes H) \longrightarrow \text{End}_A^H(H \otimes A), \quad \xi(f) = \tau f \tau^{-1},$$

then, after a short computation, we get that

$$\xi(f \cdot h^*) = \xi(f) \cdot (g \mapsto h^*).$$

Since $h^* \mapsto g \mapsto h^*$ is an automorphism of $H^*$, it follows that $\text{End}_H^H(A \otimes H) \simeq \text{End}_A^H(H \otimes A)$ as right $H^*$-modules.

As we said, the above corollary provides a new proof for Proposition 3.1.

**Corollary 3.9** If $H$ is co-Frobenius and $A/A^{coH}$ is Galois, then

$$A\# H^{\text{rat}} \simeq \text{End}(A_{A^{coH}}) \cdot H^{\text{rat}}.$$
Proof: Since the Weak Structure Theorem holds, we have that

\[
\text{End}_A^H(H \otimes A) \simeq \text{End}((H \otimes A)^{\text{co}H}_A).
\]

On the other hand, \( H \otimes A \simeq A \otimes H \) via \( h \otimes a \mapsto \sum a_0 \otimes ha_1 \) as right \( A \)-modules and right \( H \)-comodules (as in Remark 1.6), and thus we have that \((H \otimes A)^{\text{co}H} \simeq (A \otimes H)^{\text{co}H} \simeq A \). The isomorphism in the statement then follows from Corollary 3.7.

It remains to show that this gives another proof for Proposition 3.1. Indeed, the above isomorphism sends \( h^* \) to \( \rho h^* \), left multiplication by \( h^* \) restricted to the coinvariants of \( A\#H^{*\text{rat}} \). But \( (A\#H^{*\text{rat}})^{\text{co}H} = A\#t \), where \( t \) is a right integral of \( H \). We compute thus \( (a\#h^*)(b\#t) = \sum a_0\#(h^* \leftarrow b_1)t = \sum ab_0h^*(b_1g^{-1})\#t \), where \( g \) is the distinguished grouplike element from Proposition 1.9. Since this isomorphism is nothing but \( \pi \) from Proposition 3.1 composed with the automorphism of \( A\#H^{*\text{rat}} \) sending \( a\#h^* \) to \( a\#g^{-1} \mapsto h^* \), it follows that \( \pi \) is an isomorphism too.

We move next to the second duality theorem for co-Frobenius Hopf algebras. The sketch of the proof is the following:

**Step I:** Write Theorem 1.7 i) for the two-sided Hopf module \( H \otimes A \) from Example 1.5:

\[
\text{End}_A^H(H \otimes A)\#H \simeq \text{END}_A(H \otimes A).
\]

**Step II:** Define the right \( H^* \)-module structure on \( \text{END}_A(H \otimes A) \) as follows: if \( f \in \text{END}_A(H \otimes A) \) and \( f = \sum f_i e_i \) for some \( f_i \in \text{End}_A^H(H \otimes A) \) and \( e_i \in H \), then

\[
f \cdot h^* = \sum (f_i \cdot h^*)e_i
\]

(13)

**Step III:** Take the rational part on both sides of the isomorphism in Step I with respect to the \( H^* \)-module structure in Step II, and use Corollary 3.7 to get

\[
(A\#H^{*\text{rat}})\#H \simeq \text{END}_A(H \otimes A) \cdot H^{*\text{rat}}.
\]

**Step IV:** Describe \( \text{END}_A(H \otimes A) \cdot H^{*\text{rat}} \) as a matrix ring.

We are ready to prove the second duality theorem for co-Frobenius Hopf algebras.

**Theorem 3.10** ([20, Theorem 5.5]) Let \( H \) be a co-Frobenius Hopf algebra and \( A \) a right \( H \)-comodule algebra. Then

\[
(A\#H^{*\text{rat}})\#H \simeq M^I_H(A),
\]

where \( M^I_H(A) \) denotes the ring of matrices with rows and columns indexed by a basis of \( H \), and with only finitely many non-zero entries in \( A \).
Proof: It is clearly enough to prove the assertion in Step IV) above, i.e. that

\[ \text{END}_A(H \otimes A) \cdot H^{rat} \simeq M^1_A(A), \]

where the \( H^* \)-module structure is the one given by (13). We define another  \( H^* \)-module structure on \( \text{END}_A(H \otimes A) \) and show that the rational parts with respect to this module structure and the one given by (13) are the same.

Define, for \( h^* \in H^*, \ f \in \text{END}_A(H \otimes A) \)

\[ (f \circ h^*)(h \otimes a) = \sum f(h^*(S(h_1)g^{-1})h_2 \otimes a). \]

We check that \( \text{END}_A(H \otimes A) \cdot H^{rat} = \text{END}_A(H \otimes A) \circ H^{rat} \). Let \( f \in \text{END}_A(H \otimes A), \ f = \sum f_je_j, \ j \in End^H_A(H \otimes A), \ e_j \in H. \) Then

\[ (f \cdot h^*)(h \otimes a) = \left( \sum (f_j \cdot h^*)e_j \right)(h \otimes a) \]

\[ = \sum (f_j \cdot h^*)(e_jh \otimes a) \]

\[ = \sum h^*(S(e_jh_1)g^{-1})f_j(e_jh_2 \otimes a) \text{(we used (12))} \]

\[ = \sum (S(ge_jh_1)g^{-1} \rightarrow h^*)(S(h_1)g^{-1})(f_je_j)(h_2 \otimes a) \]

\[ = \sum (f_je_j) \circ (S(ge_jh_1)g^{-1} \rightarrow h^*)(h \otimes a), \]

which shows that \( \text{END}_A(H \otimes A) \cdot H^{rat} \subseteq \text{END}_A(H \otimes A) \circ H^{rat} \). Conversely, if \( f \in \text{END}_A(H \otimes A), \ f = \sum f_je_j, \ j \in End^H_A(H \otimes A), \ e_j \in H, \) then

\[ (f \circ h^*)(h \otimes a) = \left( \sum f_je_j \right)(h \otimes a) \]

\[ = \sum h^*S(gh_1)f_j(e_jh_2 \otimes a) \]

\[ = \sum h^*S(g(e_je_jh_1)f_j(e_je_jh_2 \otimes a) \]

\[ = \sum h^*S(S(ge_jh_1)g^{-1}ge_jh_1)f_j(e_je_jh_2 \otimes a) \]

\[ = \sum h^*(S(ge_jh_1)S(ge_jh_1)f_j(e_je_jh_2 \otimes a) \]

\[ = \sum (S^2(ge_jh_1)g^{-1} \rightarrow h^*)(S(e_je_jh_1))f_j(e_je_jh_2 \otimes a) \]

\[ = \sum f_j \cdot (S^2(ge_jh_1)g^{-1} \rightarrow h^*)(e_je_jh \otimes a) \]

\[ = \sum (f_j \cdot (S^2(ge_jh_1)g^{-1} \rightarrow h^*))e_j(h \otimes a), \]

showing that \( \text{END}_A(H \otimes A) \circ H^{rat} \subseteq \text{END}_A(H \otimes A) \cdot H^{rat} \).

Now write \( H = \oplus E(S_\lambda), \) where the \( S_\lambda \)'s are simple right \( H \)-comodules, and take \( \{ h_i \} \) a basis in each \( E(S_\lambda) \). Put them together and take \( \{ g^{-1}h_i \otimes 1 \} \), which is an \( A \)-basis for \( H \otimes A \). Use this basis to view the elements of \( \text{END}_A(H \otimes A) \) as row finite matrices with entries in \( A \). We show now that under this identification the elements of \( \text{END}_A(H \otimes A) \circ H^{rat} \) are represented by finite matrices. As in the proof of Corollary 3.3, we denote by \( p_\lambda \) the linear form on \( H \) equal to \( \varepsilon \) on \( E(S_\lambda) \) and 0 elsewhere. It is easy to see that for every \( f \in \)}
Lemma 1.2 shows that $S$ the right integral with a left integral, and use this bijection to identify $A$ if $A$. This shows that $\pi \circ p_\lambda S$ is represented by a finite matrix for all $\lambda$. Conversely, if $f(g^{-1}h_i \otimes 1) \neq 0$ and $f(g^{-1}h_j \otimes 1) = 0$ for all $j \neq i$, then clearly $f \in \text{END}_A(H \otimes A)$. Moreover, if $h_i \in E(S_\lambda)$, then if $h_j \in E(S_\lambda)$, we have

$$
(f \circ p_\lambda S)(g^{-1}h_j \otimes 1) = \sum p_\lambda S(gg^{-1}h_j_1)f(g^{-1}h_j_2 \otimes 1) = \sum \varepsilon(h_j_1)f(g^{-1}h_j_2 \otimes 1) = f(g^{-1}h_j \otimes 1).
$$

Since for $h_j \notin E(S_\lambda)$, $(f \circ p_\lambda S)(g^{-1}h_j \otimes 1) = f(g^{-1}h_j \otimes 1) = 0$, we get that $f = f \circ p_\lambda S \in \text{END}_A(H \otimes A) \circ H^{\text{rat}}$, and the proof is complete.

We end by giving some comments on the isomorphisms and the structures involved in the proof of the duality theorems.

Remark 3.11 [19, Lemma 1.2] shows that $\text{End}_A^H(H \otimes A) \simeq \text{Hom}_k(H, A)$ as vector spaces. Taking the embedding of $A\#H^*$ to $\#(H, A)$, and then mapping through Ulbrich’s isomorphism, we obtain the map

$$
\pi' : A\#H^* \longrightarrow \text{End}_A^H(H \otimes A), \quad \pi'(b\#h^*)(h \otimes a) = \sum h_2 S(b_2) \otimes h^*(S(h_1))b_0a.
$$

It is not hard to check that $\pi'$ is actually a ring embedding. In fact, $\pi'$ is the embedding of $A\#H^*$ composed with the isomorphism $\beta$ from Theorem 2.1 i) (see Remark 2.4 1). If $A/A^{\text{coH}}$ is Galois and $H$ is co-Frobenius, then composing the restriction of $\pi'$ to $A\#H^{\text{rat}}$ with the isomorphism $\text{End}_A^H(H \otimes A) \simeq \text{End}(A_{A^{\text{coH}}})$, we get

$$
\begin{align*}
a & \mapsto a \otimes 1 \\
& \mapsto \sum S(a_1) \otimes a_0 \\
& \mapsto \sum S(a_1)S(b_1) \otimes h^*(s(S(a_1))b_0a_0 \\
& = \sum S(b_1a_1) \otimes h^*(a_2)b_0a_0 \\
& \mapsto \sum h^*(a_2)b_0a_0 \otimes S(b_2a_2)b_1a_1 \\
& = \sum h^*(a_1)b_0a_0 \otimes 1 \\
& \mapsto \sum h^*(a_1)b_0a_0 \\
& = \pi(b\#h^*)(a).
\end{align*}
$$

This shows that $A\#H^{\text{rat}} \simeq \#(H, A) \cdot H^{\text{rat}}$. Compare this to the remark, made in [2, p. 169], that even if $A/A^{\text{coH}}$ is Galois, $A\#H^{\text{rat}}$ may be properly contained in $H^{\text{rat}} \cdot (\#(H, A))$.

Finally, we remark that if we replace in the definition of $\theta$ from Lemma 3.6 the right integral with a left integral, and use this bijection to identify $H \otimes A$ with $A\#H^{\text{rat}}$, then the map $\pi'$ above is nothing else but right multiplication.
with elements from $A\# H^*$. This explains the presence of the distinguished grouplike element in the definition of the module structure given by (12).

References


